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THE UNIVERSITY OF ALBERTA

EFFECT OF HETEROGENEITY IN TWO-PHASE FLOW SYSTEMS

by

© Yothin Tongpenyai

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY
IN
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DEPARTMENT OF MINING, METALLURGICAL, AND PETROLEUM
ENGINEERING

EDMONTON, ALBERTA
FALL, 1988

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Ramon Bentsen

Supervisor

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Date July 18th, 1988

DEDICATION

To my wife, Boonrasri, and my daughter, Aimon

ABSTRACT

It is shown that the exponential form of the autocorrelation function for log-permeability is appropriate for one-, two- and three-dimensional flow through porous media. This autocorrelation function is used to study two-phase flow through porous media.

Using spectral techniques and the exponential autocorrelation function of log-permeability, and assuming that the perturbation part of oil pressure is statistically homogeneous and isotropic, only the variance of oil pressure for a three-dimensional system could be found. Neither the variance of oil pressure for one- and two-dimensional systems, nor the covariance of oil pressure for one-, two- and three-dimensional systems could be obtained.

An expression for the expected value of sample covariance is derived for the generated values of a variate which has an exponential autocorrelation function. Moreover, it is demonstrated that sample variance, covariance, and integral scale, which are obtained from generated values of a variate which is correlated, are biased. In addition, it is shown that the expression for the expected value of sample covariance can be used to justify the validity of the generated values of a variate used in Monte Carlo techniques.

A Monte Carlo study of steady-state, two-phase flow in a bounded domain shows that the standard deviation of log-permeability seems to be a more important parameter than the integral scale of log-permeability. In addition, heterogeneity in rock properties has an insignificant effect on the fractional flow of water at any given saturation. Moreover, mobility ratio and capillary pressure have an insignificant influence on the effect of heterogeneity in rock properties on the resulting pressure and saturation.

A method to estimate the average effective permeability to each phase in a heterogeneous system is proposed. When investigating the concept of effective permeability, and when studying the effect of heterogeneity of rock properties on the resulting pressure and saturation, how rate and pressure gradient are treated is important.

A Monte Carlo study on transient two-phase flow in a bounded domain shows that the effect of heterogeneity of rock properties is more pronounced in the frontal region than in the regions ahead of or behind the frontal region.

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Table of Contents

Chapter Page

I. INTRODUCTION 1

II. LITERATURE SURVEY 7

A. One-phase flow problems 8

1. Spatially uncorrelated parameters and bounded systems 8

2. Spatially correlated parameters and bounded systems 10

3. Spatially correlated parameters and unbounded systems 13

4. Effective permeability 16

5. Other related work 19

B. Two-phase Flow Problems 20

III. STOCHASTIC METHODS 23

A. Distribution function and probability density function 23

B. Mean, variance, covariance, and correlation function 24

C. Integral scale 28

D. Statistically homogeneous and statistically isotropic 30

E. Stochastic methods 34

IV. ASSUMPTIONS AND MATHEMATICAL FORMULATION 37

A. Scale 37

B. Assumptions concerning the flow condition 39

C. Probability distributions of permeability and porosity 39

D. Exponential autocorrelation for log-permeability 40

E. Spectral techniques 48

V. MONTE CARLO SIMULATION TECHNIQUES 61

A. Generation method 62

B. Random number generator 67

C. Effect of correlation 69

D. Generation of correlated permeability and porosity 92

VI. NUMERICAL MODEL 97

A. Grid system 98

B. Initial and boundary conditions 98

C. Flow conditions 100

D. Relationship for relative permeabilities 101

E. Relationship for irreducible water saturation 104

F. Verification of the simulator 107

VII. STEADY-STATE FLOW 112

A. Data 112

B. Constant flow rate and constant pressure at the outlet end 117

Effect of σ_Y and λ_Y' on the mean oil pressure 117

Effect of σ_Y and λ_Y' on the mean water saturation 132

Effective permeability in two-phase flow problems 132

Effect of σ_Y and λ_Y' on the standard deviation of oil pressure 139

Effect of σ_Y and λ_Y' on the standard deviation of water saturation 148

Effect of fractional flow 150

Effect of mobility ratio 155

Effect of capillary pressure 155

Effect on plot of fractional flow versus mean of water saturation 159

C. Constant pressure at both ends 159

Effect of σ_Y and λ'_Y on the mean oil pressure 161

Effect of σ_Y and λ'_Y on the mean flow rate .1[illegible]

Effect of σ_Y and λ'_Y on the mean of water saturation 168

Effect of σ_Y and λ'_Y on the standard deviation of oil pressure 168

Effect of σ_Y and λ'_Y on the standard deviation of oil flow rate 170

Effect of σ_Y and λ'_Y on the standard deviation of water saturation 175

Effect of fractional flow 175

Effect of mobility ratio and capillary pressure 179

D. Significance of problem formulation 179

E. Implication of the results 180

F. Estimate of relative permeability 182

G. Test of independence and normality 191

H. Regression analysis of oil pressure difference 196

VIII. TRANSIENT FLOW 201

A. Effect of σ_Y and λ'_Y on the mean of oil pressure 201

B. Effect of σ_Y and λ'_Y on the standard deviation of oil pressure 204

C. Effect of σ_Y and λ'_Y on the mean of water saturation 204

D. Effect of σ_Y and λ'_Y on the standard deviation of water saturation 204

E. Effect of mobility ratio and capillary pressure 209

F. Implication of the results 212

IX. CONCLUSIONS AND RECOMMENDATIONS 214

REFERENCES 221

APPENDIX A 234

A. Effective permeability for one-dimensional, parallel systems 234

B. Effective permeability for one-dimensional, serial systems 238

APPENDIX B 241

A. Covariance function for one-dimensional systems 241

B. Covariance function for two-dimensional systems 242

C. Covariance function for three-dimensional systems 242

APPENDIX C 244

A. Variance of oil pressure for one-dimensional systems 244

B. Variance of oil pressure for two-dimensional systems 246

C. Variance of oil pressure for three-dimensional systems 247

APPENDIX D 250

APPENDIX E 260

List of Tables

Table | Page

V-1 Correction terms for sample autocovariance: $\lambda/L = 0.05$ 75

V-2 Correction terms for sample autocovariance: $\lambda/L = 0.02$ 76

V-3 Correction terms for sample autocovariance: $\lambda/L = 0.5$ 77

V-4 Correction terms for sample autocovariance: $\lambda/L = 0.8$ 78

V-5 Analytical values of sample integral scale ratio 80

V-6 Analytical and observed values of sample integral scale ratio for different numbers of blocks 86

V-7 Analytical and observed values of sample integral scale ratio for different values of input integral-scale ratio 86

V-8 Test results for the case with $\sigma_Y = 0.8$, $\lambda_Y' = 0.3$ 90

VII-1 Values of standard deviation and integral scale ratio of log-permeability used in the Monte Carlo runs 113

VII-2 Comparison of input and overall means of log-permeability 124

VII-3 Comparison of the experimental and the theoretical values of effective permeability 126

VII-4 Effect of number of realizations on % deviation from the straight line 130

VII-5 Maximum and minimum values of $\bar{S}_w$ for steady-state flow 133

VII-6 Estimated mean of $\bar{S}_w$ for steady-state flow 134

VII-7 Comparison of effective permeability to oil from the numerical experiments and from the relationship, Eq. (VII-5) 136

Table Page

VII-8 Values of coefficient γ 145

VII-9 Effective permeability to oil for cases with constant pressure at both ends 165

VII-10 Mean, maximum and minimum values of $\bar{S}_w$ along the system (steady-state flow, constant pressure at both ends) 169

VII-11 Number of cores needed for 1% allowable error in estimating mean pressure 188

VII-12 Number of cores needed for 5% allowable error in estimating mean pressure 189

VIII-1 Data for transient flow 202

List of Figures

Figure Page

IV-1 Autocorrelation function for ln(k) from 8 boreholes in the Mt. Simon aquifer compared with exponential function 49

V-1 Difference between input and sample integral scale 82

V-2 Porosity versus permeability of conventional core plugs from Miocene sandstones, Pattani basin, Gulf of Thailand 93

V-3 Core porosity versus permeability of North Sea sandstones 94

V-4 Porosity-permeability plot by zone, Prudhoe Bay 95

VI-1 The point-distributed grid system used in this study 99

VI-2 Typical relationship between relative permeability and water saturation used in this study 105

VI-3 Typical capillary pressure curve for imbibition processes used in this study 106

VI-4 Comparison of numerical and analytical results using linear relationship between relative permeability and saturation 108

VI-5 Saturation profile from numerical method using Eqs. (VI-7) and (VI-8) for relationship between relative permeability and saturation 109

VI-6 Comparison of saturation profile for different number of blocks 111

VII-1 Examples of realization for different values of standard deviation of log-permeability 115

VII-2 Examples of realization for different values of integral scale of log-permeability 116

VII-3 Effect of σ_Y on mean oil pressure, $\lambda_Y' =$ 0.1 118

Figure Page

VII-4 Effect of λ'_Y on mean oil pressure, σ_Y = 0.5 119

VII-5 Effect of σ_Y and λ'_Y on mean oil pressure 120

VII-6 Effect of σ_Y and λ'_Y on mean oil pressure gradient for single-phase flow (based on the inlet and outlet end pressures) 122

VII-7 Effect of σ_Y and λ'_Y on mean pressure for single-phase flow 128

VII-8 Effect of number of realizations for highly heteogeneous system (λ'_Y = 0.3, σ_Y = 0.8) 129

VII-9 Effect of σ_Y on standard deviation of oil pressure 140

VII-10 Effect of λ'_Y on standard deviation of oil pressure 142

VII-11 Effect of σ_Y and λ'_Y on standard deviation of oil pressure 143

VII-12 Effect of σ_Y and λ'_Y on standard deviation of water saturation 149

VII-13 Effect of f_w on $\bar{p}_o$, $\sigma_Y = 0.5$, $\lambda'_Y = 0.1$ 151

VII-14 Effect of f_w on s_{p_o}, $\sigma_Y = 0.5$, $\lambda'_Y = 0.1$ 152

VII-15 Effect of f_w on $\bar{S}_w$, $\sigma_Y = 0.5$, $\lambda'_Y = 0.1$ ($\mu_o/\mu_w = 5$, $P_c = 0$) 153

VII-16 Effect of f_w on s_{S_w}, $\sigma_Y = 0.5$, $\lambda'_Y = 0.1$ ($\mu_o/\mu_w = 5$, $P_c = 0$) 154

VII-17 Effect of σ_Y and λ'_Y on mean of oil pressure (favourable cases) 156

VII-18 Effect of σ_Y and λ'_Y on standard deviation of oil pressure (favourable cases) 157

VII-19 Effect of σ_Y and λ'_Y on standard deviation of water saturation (favourable cases) 158

VII-20 Effect of σ_Y and λ'_Y on fractional flow curve 160

Figure Page

VII-21 Effect of σ_Y and λ'_Y on mean oil pressure (constant pressure at both ends) 162

VII-22 Effect of σ_Y and λ'_Y on mean of oil flow rate 163

VII-23 Effect of σ_Y on standard deviation of oil pressure (constant pressure at both ends) 171

VII-24 Effect of λ'_Y on standard deviation of oil pressure (constant pressure at both ends) 172

VII-25 Effect of σ_Y and λ'_Y on standard deviation of oil pressure (constant pressure at both ends) 173

VII-26 Effect of σ_Y and λ'_Y on standard deviation of oil flow rate 174

VII-27 Effect of σ_Y and λ'_Y on standard deviation of water saturation (constant pressure at both ends) 176

VII-28 Effect of fractional flow on standard deviation of oil pressure (constant pressure at both ends) 177

VII-29 Effect of fractional flow on standard deviation of water saturation (constant pressure at both ends) 178

VII-30 The stabilized probability plot for oil pressure of steady-state flow with constant rate, $\lambda'_Y = 0.3$, $\sigma_Y = 0.2$ 194

VII-31 The stabilized probability plot for oil pressure of steady-state flow with constant rate, $\lambda'_Y = 0.3$, $\sigma_Y = 0.8$ 195

VIII-1 Effect of σ_Y and λ'_Y on mean of oil pressure (transient flow) 203

VIII-2 Effect of σ_Y and λ'_Y on standard deviation of oil pressure (transient flow) 205

VIII-3 Effect of σ_Y on mean of water saturation (transient flow) 206

Figure Page

VIII-4 Effect of λ'_Y on mean of water saturation (transient flow) 207

VIII-5 Effect of σ_Y and λ'_Y on standard deviation of water saturation (transient flow) 208

VIII-6 Mean saturation profile and one standard deviation interval, $\sigma_Y = 0.2$, $\lambda'_Y = 0.05$ 210

VIII-7 Mean saturation profile and one standard deviation interval, $\sigma_Y = 0.8$, $\lambda'_Y = 0.3$ 211

A-1 Flow in a parallel system 235

A-2 Flow in a serial system 235

NOMENCLATURE

A	area, L^2
a	a constant in the equation expressing the relation between log-permeability and porosity
a	a constant used in a congruential generator
b	a constant in the equation expressing the relation between log-permeability and porosity
b_0, b_1, b_2, b_3, b_4	estimates of parameters of a model for regression analysis
C_{ff}	autocovariance function of f
C(k)	autocovariance of lag k
C_{pp}	autocovariance function of p, $(m/Lt^2)^2$
C_{XX}	autocovariance of random variable X
C_{XY}	crosscovariance of random variables X and Y
$C_{X,p}$	autocovariance of random variable X with p dimensions
c	a constant used in the congruential generator
c(k)	sample autocovariance
D	depth, L
E	maximum allowable error
E{ }	the expectation operator
e or exp	exponential
F	the mean of ln(k)
F( )	distribution function
f	random (or perturbation) part of ln(k)
f( )	probability function

$f^*(x')$	conjugate of $f(x)$
f_w	fractional flow = q_w/q
g	acceleration of gravity, L/t^2
$g(x)$	any function
H	thickness of the system, L
h	scalar distance, L
$\underline{h}$	a vector between any two points
$\underline{I}$	identity matrix
i	$\sqrt{-1}$ (in the section involving spectral techniques)
J	mean pressure gradient in one-dimensional flow, m/L^2t^2
J_1	mean pressure gradient in the main flow direction, m/L^2t^2
$J_r(t)$	Bessel function of order r
$\underline{K}$	covariance matrix
$K_r(t)$	modified Bessel function of the second kind of order r
k	permeability, L^2, or any integer
k^*	the permeability of the system, L^2
k_{eff}	effective permeability, L^2
$k_{eff,abs}$	effective (absolute) permeability, L^2
$k_{eff,est}$	estimated effective permeability, L^2
$k_{eff,min}$	lower bound of effective permeability, L^2
$k_{eff,max}$	upper bound of effective permeability, L^2
$k_{eff,o}$	effective permeability to oil, L^2
k_G	geometric mean of permeability, L^2
k_i	permeability of realization i, L^2

k_{ij}	permeability for block j and realization i, L^2
k_{oiw}	effective permeability to oil at irreducible water saturation, L^2
k_{ro}	relative permeability to oil
$\bar{k}_{ro}$	relative permeability to oil evaluated at the mean value of S_w (or average relative permeability to oil)
k_{rw}	relative permeability to water
k_{wor}	effective permeability to water at residual oil saturation, L^2
L	length of a system, L
$\underline{L}$	lower triangular matrix
$\underline{L}'$	transpose of $\underline{L}$
m	any integer
m	constant used in a congruential generator
$\underline{m}$	column vector of specified mean
n	any integer
P( )	probability
P	mean of p_o, m/Lt^2
P_c	capillary pressure, m/Lt^2
p	pressure, m/Lt^2, random (or perturbation) part of oil pressure, m/Lt^2, or number of dimension
p_o	oil pressure, m/Lt^2
$\bar{p}_o$	mean oil pressure, m/Lt^2
p_w	water pressure, m/Lt^2
$\bar{p}_w$	mean water pressure, m/Lt^2
q	flow rate or total flow rate, L^3/t
q_o	oil flow rate, L^3/t

$\bar{q}_o$	mean oil flow rate, L^3/t
q_o^*	injected or produced oil flow rate, L^3/t
q_{o1}	oil flow rate in the main flow direction, L^3/t
q_w	water flow rate, L^3/t
q_w^*	injected or produced water flow rate, L^3/t
R	continuous domain ($-\infty$ to $+\infty$)
RF	recovery factor
S_{ff}	spectral density function of f
S_o	oil saturation
S_{pp}	spectral density function of p
S_w	water saturation
$\bar{S}_w$	mean water saturation
S_w^*	normalized saturation
$S_{w,dr}^*$	normalized saturation for drainage process
$S_{w,imb}^*$	normalized saturation for imbibition process
S_{wirr}	irreducible water saturation
$S_{X,p}$	the spectral density function of X with p dimensions
$\underline{s}$	position vector for point s
$\underline{s}_1$	position vector for point s_1
$\underline{s}_2$	position vector for point s_2
$\underline{s}_n$	position vector for point s_n
s^2	variance of a sample
s_{p_o}	standard deviation of oil pressure, m/Lt^2
s_{q_o}	standard deviation of oil flow rate, L^3/t
s_{S_w}	standard deviation of water saturation
t	time variable, t

$\mathrm{Var}\{\ \}$	the variance operator
$\underline{w}$	an independent normally distributed random number with zero mean and unit standard deviation
$\underline{w}'$	transpose of $\underline{w}$
X	a random variable X
$X_1, .., X_n$	input random variables
X_i	a random variable X at point i
X_{i-1}	a random variable X at point i-1
X_{i+1}	a random variable X at point i+1
X_n, X_{n+1}	random numbers
x	a realization of X or x-coordinate
x'	conjugate of x
$\bar{x}$	sample mean
Y	a random variable Y or ln(k)
Y_{ij}	ln(k) for block j and realization i
$\bar{Y}_{overall}$	overall mean value of Y (evaluated from a sample)
y	a realization of Y or y-coordinate
$\underline{Y}$	a random vector
Z	$\log_{10}(k)$ or output random variable
Z_f	the spectral function for f
Z_p	the spectral function for p
Z_f^*	conjugate of Z_f
Z_p^*	conjugate of Z_p
$z_{1-\alpha/2}$	the $100(1-\alpha/2)$ percent point of a standard normal distribution.
α^2	the coefficient in the second order differential equation derived from the nearest-neighbour scheme

β	a parameter for the nearest-neighbour scheme
$\Gamma(t)$	the gamma function
γ	coefficient
2γ	variogram
ΔL	distance, L
Δp	pressure difference, m/Lt^2
Δx_i	the length of layer i, L
Δz_i	the thickness of layer i, L
$\delta(\omega-\omega')$	the Dirac delta function
δ_i	a white-noise function consisting of uncorrelated values (in a simple autoregression process)
ϵ	a multiplicative random error
η_i	an independent normal random variable with zero mean and variance, σ_η, located at point i.
θ_1, θ_2	angles
λ	integral scale
λ_f	integral scale of f
λ_X	integral scale of random variable X
λ_Y	integral scale of ln(k)
λ'	integral scale ratio = λ/L
λ'_Y	integral scale ratio of ln(k) = λ_Y/L
$\hat{\lambda}$	sample integral scale
$\hat{\lambda}_{anal}$	analytical value of the sample integral scale
$\hat{\lambda}'_{anal}$	analytical value of the sample integral-scale ratio
$\hat{\lambda}'_{obs}$	observed value of the sample integral-scale ratio
μ	mean of a population or viscosity, m/Lt

μ_o	oil viscosity, m/Lt
μ_w	water viscosity, m/Lt
μ_X	mean of random variable X
μ_Y	mean of random variable Y or log(k)
μ_ϕ	mean of porosity
ρ	autocorrelation function of lag one
ρ_k	autocorrelation function of lag k
ρ_o	oil density, m/L^3
ρ_w	water density, m/L^3
ρ_{XX}	autocorrelation function of random variable X
ρ_{XY}	cross-correlation function of random variables X and Y
$\rho_{X,p}$	autocorrelation function of X with p dimensions
$\sum$	summation
σ	standard deviation of a population
σ'	initial estimate of the process standard deviation
σ_X	standard deviation of random variable X
σ_Y	standard deviation of random variable Y or of ln(k)
σ_ϕ	standard deviation of porosity
σ^2	variance of a population
σ_f^2	variance of f
σ_p^2	variance of p
σ_x^2	variance of random variable X
σ_Y^2	variance of random variable Y
σ_η^2	variance of random variable η
ϕ	porosity

ω	frequency (in Fourier or frequency space)
$\underline{\omega}$	vector quantity of ω
$\omega_1, \omega_2, \omega_3$	elements in vector $\underline{\omega}$
∇^2	the second-order partial derivative operator

I. INTRODUCTION

In solving flow equations, two types of methods can be used: deterministic methods and stochastic methods. When using deterministic methods, all variables and parameters are treated as deterministic variables. That is, these variables and parameters are functions of time and/or space only and can be predicted exactly, if all the necessary information is known. The conventional methods of solving flow equations in petroleum engineering are deterministic methods. On the other hand, when using stochastic methods the variables and/or parameters are treated as random variables. These random variables are functions of time and/or space and are also a function of elements or points in sample space. The sample space is the set of all possible outcomes of an experiment or the set of all possible values that variables or parameters can possess.[1] When variables and/or parameters are treated as random variables, it is not possible to predict the exact values of the output variables using the flow equations at any specific time and place.

The logical question to be raised at this point is why is it necessary to treat variables and parameters in the flow equations as random variables given the problems which arise out of such an approach? The reason is that it is not possible to supply all the information about rock properties (or even fluid properties and rock-fluid interaction properties with which we will not be concerned here) when the deterministic approach is taken. Partial knowledge about (or uncertainty in values of) rock properties, such as permeability and porosity, leads to uncertainty in predictions of pressure and saturation values at any specific time and place. Uncertainty in rock properties is a rule rather than an exception in petroleum engineering. Therefore, it is appropriate to treat variables and parameters in the flow equations as random variables, or in other words, treat the flow equations as stochastic (or random) differential equations.

The philosophical concept behind treating the flow equations as stochastic differential equations, because of partial or incomplete knowledge about rock properties, is that flow through porous media is a deterministic phenomenon which can be described (or predicted) exactly by the flow equations. However, due to incomplete knowledge (because of either inability or lack of interest in obtaining it) about rock properties, the dependent variables, e.g. pressure and saturation, cannot be predicted exactly. This philosophical approach is taken because the partial differential equations governing flow in porous media are usually available. This approach has already been taken in the field of turbulent flow as aptly stated by Lumley[2] in one of the footnotes of his book:

> "The underlying philosophical concept is that there is a completely deterministic mechanism, and one simply lacks information about it and its inputs; to a quantum mechanicist, this view point is anathema. In fact, the matter is philosophical, since it is not possible to settle by observation whether the

> mechanism in question is deterministic but disturbed by influences about which no information is available, or is in some sense not deterministic. Since we will always be in possession of a governing differential equation the former point of view is more useful to us."

The results from stochastic methods are reported in terms of mean and standard deviation or its square, variance. The mean is, loosely speaking, an arithmetic average of all possible outcomes of the result, say pressure, while the standard deviation is an indicator of how all possible values of the result fluctuate around the mean. (Formal definitions of mean and standard deviation will be given later.) The mean and standard deviation values of the result will reflect the degree of heterogeneity of rock properties, which can be expressed in terms of standard deviation or variance, auto- and cross-covariances, and integral scales of permeability and porosity. By studying the effects of these statistics on the mean and standard deviation values of the result, one can learn more about the effect of heterogeneity on flow through porous media.

Using stochastic methods, only simple cases can be solved analytically. The more complicated cases will have to be solved numerically. This can be prohibitively expensive. Therefore, it seems that solving the flow equations by stochastic methods will be appropriately a supplement to solving them by deterministic methods. That is, while the deterministic methods are used routinely in solving the flow equations, the stochastic methods will help in justifying the results of the deterministic methods. First, the stochastic results can be used to justify the use of deterministic methods. It should be noted that in every application of conventional deterministic methods some kind of averaging (at the macroscopic scale) has been used implicitly. This is because of incomplete knowledge of the rock properties. For example, in well test analysis it is assumed that the rock properties around the well can be represented by a single fictitious block with uniform properties. In solving the flow equations numerically, it is also necessary to assume that the properties of the rock within each simulation block can be represented by a single fictitious block with uniform properties. Using stochastic methods will help to justify these implicit assumptions.

Second, because the application of conventional deterministic methods implicitly implies the use of some kind of averaging, it is illogical to expect, for example, that the predicted (or calculated) pressures should match exactly the field pressures. The results from the stochastic methods will provide information about how far the predicted pressures will deviate from the field pressures, for the given properties of the porous media. This will be very useful in the history-matching process and will prevent simulation operators from trying to overmatch the pressure or other variable values which may lead to unreasonable adjusted rock-property values.

It has always been thought that using stochastic or statistical methods will require much more data than using deterministic methods. This is true if one refers to the conventional deterministic methods which implicitly use some kind of averaging at the macroscopic scale. The application of actual deterministic methods, i.e. no averaging of any kind involved, will require a detailed knowledge of rock properties at a scale much smaller than a region around the well or a simulation block. It is in this sense that the deterministic methods require much more data than the stochastic methods.

From the reasoning above it can be seen that the stochastic methods are very useful tools in solving the flow equations. They can be used as routine tools (if it is permissible) or as supplemental tools to the deterministic methods.

The objective of this study is to investigate the effect of variability in permeability on the resulting pressure and saturation of two-phase flow through porous media. It was decided that:

1. spectral techniques and Monte Carlo simulations would be used for this purpose,
2. steady-state flows through one-, two-, and three-dimensional systems would be considered employing spectral techniques,
3. in the Monte Carlo study, only the one-dimensional system would be used and both steady-state and transient flows would be considered,
4. influence of mobility ratio and capillary pressure on

the effect of heterogeneity in rock properties would also be investigated, and

5. the procedure to obtain the average relationship between relative permeabilityies and saturation would be discussed.

II. LITERATURE SURVEY

The pioneer work on the application of stochastic techniques to flow through porous media seems to be the work by Warren and Price.[3] The other important study is the one by Freeze.[4] This latter study initiates systematic research on the application of stochastic techniques to groundwater hydrology. A comprehensive review on this subject up to 1980 was undertaken by Neuman.[5] Another theoretical review is the one by Gelhar.[6] More recent reviews are presented by Dagan[7] and Gelhar.[8] In the following, application of stochastic techniques to one-phase flow problems will be reviewed first. The review will start with the bounded systems. Then the unbounded systems will be discussed. After that, the concept of effective permeability will be reviewed. Finally, related works, such as the application of geostatistical methods and the application of stochastic methods to inverse problems for one-phase flow will be mentioned. Utilization of stochastic techniques for dispersion problems (e.g., Refs. 9-16), which is another important research area, will not be discussed here. The application of stochastic techniques to problems of two-phase flow in porous media, which is the main concern in this study, will be discussed after reviewing the one-phase flow problems.

A. One-phase flow problems

1. Spatially uncorrelated parameters and bounded systems

The first systematic study of flow in heterogeneous porous media using stochastic techniques was undertaken by Warren and Price.[3] They studied the effect of permeability variation on both steady-state and transient flow in three-dimensional systems. They assumed that there was no spatial correlation of permeability in the system. Monte Carlo techniques were used in their study. Based on computational experiments and subject to the limitations implied by the assumptions made, Warren and Price concluded that: 1. the most probable behaviour of a heterogeneous system with single-phase flow approaches that of a homogeneous system having a permeability equal to the geometric mean of the individual permeabilities; 2. the effects of flow geometry and anisotropy on the most probable value for the effective permeability of a heterogeneous medium are finite but not significant; and 3. the permeability determined from a pressure build-up test of a heterogeneous reservoir gives a reasonable value for the effective permeability within the drainage area.

Freeze[4] utilized Monte Carlo techniques to solve one-phase flow through one-dimensional bounded (serial) systems. He took into account mutual correlation of conductivity, compressibility, and porosity. However, he assumed that there were no spatial correlations of these parameters. Freeze assumed that conductivity and compressibility were log-normally distributed and that porosity was normally distributed. Freeze considered two situations: steady-state flow under constant pressure boundary conditions (at the two ends of a serial system), and transient flow through serial systems with similar boundary conditions and prescribed pressure at time zero. In solving the flow equations, an analytical solution was used for the steady-state flow case while a numerical solution was used for the transient flow case. From the results of his work, Freeze questioned the validity of the deterministic philosophy traditionally shared by most subsurface hydrologists and reservoir engineers. He concluded that: 1. there may be no way to define an equivalent medium for transient flow in nonuniform materials; 2. the consideration of spatial variability in a stochastic sense may lead to very large fluctuations in the computed values of hydraulic head; and 3. the values of hydraulic head computed with the aid of deterministic models, whether analytical or numerical, may often suffer from very large errors.

Because his results led to some far-reaching conclusions with regard to the validity of the deterministic philosophy, Freeze[4] cautiously stated two major limitations that may have affected the universality of his conclusions. These were assumptions of spatial independence of porous medium properties and the use of a one-dimensional system in his analyses. He suspected that these two considerations could have reduced the overall range of output uncertainty. In fact, from later studies[17-21], it has been found that taking into consideration the spatial correlation of conductivity or permeability leads to a wider range of output uncertainty than not doing so. However, the one-dimensional aspect of Freeze's analysis is the main factor that causes high uncertainty of the output. As has been shown by many later investigators, analyses under more realistic two- and three-dimensional conditions lead to a smaller range of output uncertainty. Nevertheless, Freeze's conclusions had a strong positive impact on hydrologists by making them acutely aware of the potential dangers involved in regarding the subsurface environment from a purely deterministic point of view.[5]

2. Spatially correlated parameters and bounded systems

Smith and Freeze[20,21] used Monte Carlo simulation techniques to study one- and two-dimensional steady state groundwater flow in a bounded domain. The flow domain was divided into a finite set of discrete blocks. Hydraulic conductivity values in neighboring blocks were autocorrelated by assuming that spatial variations in conductivity could be represented by a first-order, nearest-neighbor stochastic model. An integral scale (defined in the next chapter) was used to characterize the average distance over which conductivity values in the block system were correlated. They concluded that: 1. the standard deviation in hydraulic head increases with an increase in either the standard deviation in hydraulic conductivity or the strength of the correlation between neighboring conductivity values for both one- and two-dimensional systems; 2. the standard deviation in head does not significantly depend upon the block size or the total number of blocks included along the flow line, provided a sufficiently accurate representation of the autocorrelation function is made; 3. the standard deviation of hydraulic head is approximately halved in the bounded two-dimensional analog of the one-dimensional flow problem; 4. the uncertainty in model predictions depends upon both the spatial heterogeneity in hydraulic conductivity and the nature of the flow system operating within the medium; 5. the uncertainties in the predicted hydraulic head values are strongly influenced by the presence of a spatial trend in the mean hydraulic conductivity; 6. in evaluating the concept of an effective conductivity for a heterogeneous medium, both the nature of the spatial heterogeneity in hydraulic conductivity and the flow system operating within the domain must be considered.

Concerning the conclusion about the magnitude of standard deviation of hydraulic head for the case of two-dimensional uniform flow in a bounded domain which is about half as large as that for a one-dimensional domain, the reduction in standard deviation of hydraulic head is smaller than the one obtained by spectral techniques.[22,23] Smith and Freeze[21] suspected that the smaller reductions in standard deviation of hydraulic head in the Monte Carlo analysis were due to the bounded nature of the conductivity realizations and to statistical nonhomogeneity in hydraulic head due to the use of constant head values on the boundaries of the flow domain.

Later, Gutjahr and Gelhar[24] studied analytically single-phase flow in bounded or finite domains. They compared the results from unbounded (or infinite) systems with those from bounded systems. They also studied the case where the hydraulic head is not stationary or statistically homogeneous. In this study they utilized three techniques: analytical approximations using first-order analysis (propagation of error), covariance differential equations, and variogram analysis. All these techniques yield consistent results which demonstrate the important effects of boundary conditions. They also found that in the one-dimensional case the head variance was very sensitive to the form of the autocorrelation function for conductivity and to the boundary conditions used. They have shown that in the one-dimensional case the stochastic process of the head is nonstationary when the integral scale of the input (conductivity) is nonzero and the variogram (see Chapter III) of the head is linear at large separation values. In the three-dimensional case it was found that the stochastic process of the head is stationary under very general conditions. Finally, they emphasized that, for the one-dimensional problem, several analytical techniques (first-order analysis, covariance differential equations, and variograms) produce consistent theoretical results which are in agreement with Monte Carlo simulations, thus demonstrating the validity and equivalence of these methodologies.

3. Spatially correlated parameters and unbounded systems

Many studies[17,18,23-29] related to the investigation of single-phase flow in unbounded domains have appeared in the literature. Gelhar and his coworkers[17,18,23-25] used spectral techniques, while Dagan[26-29] used the self-consistent approximation concept and perturbation approximations using Green function methods. In all of these studies, the variation of conductivity or permeability in porous materials is represented in a continuum sense as a spatial stochastic process which is characterized by a covariance function.

Bakr *et al.*[18] studied the effect of spatial variability of conductivity on flows through one- and three-dimensional systems. They assumed that both log-conductivity and hydraulic head are stationary (or statistically homogeneous). They showed that the head variance produced by three-dimensional statistical isotropic conductivity perturbations is only 5% of that in the corresponding one-dimensional case. Bakr *et al.* also showed that, although the head process can be assumed to be statistically homogeneous, it is anisotropic. Fluctuations in head in the direction perpendicular to the direction of mean flow have consistently higher correlation values than in any other direction for the same value of integral scale; the lowest correlation occurs in the direction of the mean flow. When comparing flows in one-and three-dimensional systems, Bakr *et al*. utilized different functional forms for the correlation function for log-conductivity. The criterion for relating the correlation distances of these two systems is a reasonable agreement between the two autocorrelation functions on the plot of autocorrelation function versus a ratio of distance to integral scale (for a three-dimensional system).

In their second paper, Gutjahr *et al*.[25] compared the exact solution of the one-dimensional flow equation and the logarithmic approximation for the case where log-conductivity is normally distributed. They concluded that the logarithmic solution is adequate for small values of the standard deviation of log-conductivity and the error increases rapidly for standard deviations greater than unity. Concerning the allowable magnitude of standard deviation of conductivity, Gutjahr[30] showed that under the assumptions that log-conductivity and head are second-order stationary jointly Gaussian processes, the covariance relationships (between those of head and conductivity), derived by using first-order approximation, are exact solutions for the nonlinearized equations, no matter how large the conductivity variance is. Dagan,[31] based on his results, indicated that this is not the case. However, he showed that the first-order approximation is very robust. Using spectral methods, Dagan demonstrated the robustness of the first-order approximation by showing that second-order corrections of the head covariance were small. He derived these second-order corrections explicitly. Gutjahr *et al.*[25] also presented (approximate) expressions for the effective conductivity for one-, two- and three-dimensional systems.

Using the self-consistent approximation concept, Dagan[26] considered effective permeability, as well as variances of head gradient, specific discharge, and head for one-, two-, and three-dimensional flows. He showed that the head variance is grossly overestimated for one-dimensional flow through blocks in series, and that the same is true for the specific discharge for layers in parallel. The two- and three-dimensional head variances are much lower and are closer to each other. This reduction of head variance when proceeding from one- to two- or three-dimensional systems is consistent with the conclusions of Gelhar,[17] Bakr *et al.*,[18] and Smith and Freeze.[21] Later Dagan[27] refined his previous work using a simplified model consisting of a collection of spherical (in three dimensions) and cylindrical (in two dimensions) blocks of different conductivity and diameter set randomly and independently in space. In this paper he investigated the steady-state flow problem. He considered the effect of the boundary on the head variance and showed that it increased with distance from the boundary. This result is consistent with that of Smith and Freeze[20,21] who used Monte Carlo simulation techniques. He also concluded that the head fluctuations field is statistically homogeneous in three dimensions, but this is not the case in two dimensions. Gutjahr and Gelhar[24] and Mizell *et al.*[23] later obtained a similar result. Although the head is not stationary (or statistically homogeneous) in two-dimensional domains, it was shown by Chirlin and Dagan[32] that in a two-dimensional domain it has stationary increments, and that the head variogram can be determined for an unbounded domain.

4. Effective permeability

One of the major questions asked in reservoir engineering is: what value of permeability should be used for an element (a unit under consideration), e.g. a simulation block, a region around a well? In other words, what kind of averaging should be used for any particular system? Many investigators have tried to answer this question. In petroleum engineering, Warren and Price[3] seem to be the first to have studied this problem systematically. They called a single-value permeability which can be used to represent a heterogeneous system the effective permeability (some investigators use the word, representative permeability). The terminology used by Warren and Price will be adopted in this study. They concluded that the most probable behaviour of a heterogeneous system approaches that of a homogeneous system with a permeability equal to the geometric mean of the individual permeabilities. These authors also stated that the permeability determined from a pressure buildup curve for a heterogeneous reservoir gives a reasonable value for the effective permeability of the drainage area.

It should be noted that Warren and Price[3] used three-dimensional systems and small values of variance of permeability (or, preferably, log-permeability). For one-dimensional systems, the lower and upper bounds of effective permeability are (see Refs. 33,34)

$$k_{eff,min} = \left[\int_{-\infty}^{\infty} k^{-1} f(k)\,dk\right]^{-1}, \qquad \text{(II-1)}$$

and

$$k_{eff,max} = \int_{-\infty}^{\infty} k f(k)\,dk, \qquad \text{(II-2)}$$

where it is assumed that permeability is statistically homogeneous. If permability is log-normally distributed, Eqs. (II-1) and (II-2) become (see Appendix A)

$$k_{eff,min} = k_G \exp(-\tfrac{1}{2}\sigma_Y^2), \qquad \text{(II-3)}$$

and

$$k_{eff,max} = k_G \exp(\tfrac{1}{2}\sigma_Y^2), \quad \text{(II-4)}$$

where

$$k_G = \exp(\mu_Y),$$

and μ_Y and σ_Y are the mean and standard deviation of log-permeability. In fact, Eq. (II-3) is the expression for the effective permeability of a serial system, whereas the expression in Eq. (II-4) is the effective permeability of a parallel system.

Gutjahr *et al.*[25] used an approximate analysis in which they linearized the logarithm of the hydraulic conductivity and obtained, for the expression for the effective permeability of serial systems,

$$k_{eff} = k_G(1 - \frac{1}{2}\sigma_Y^2), \quad \text{(II-5)}$$

This solution agrees with Eq. (II-3) to the first order in σ_Y. Eq. (II-5) underestimates the exact value of k_{eff}, defined by Eq. (II-3), by about 18% when $\sigma_Y = 1$; but when $\sigma_Y = 0.5$, the difference is less than 1%. Gutjahr *et al.*[25] also derived the expression for effective permeability for two- and three-dimensional systems, which are

$$k_{eff} = k_G, \quad \text{(II-6)}$$

and

$$k_{eff} = k_G(1 + \frac{1}{6}\sigma_Y^2), \tag{II-7}$$

respectively. Eqs. (II-5) to (II-7) were derived assuming that the logarithm of hydraulic conductivity can be linearized and the mean pressure gradient is uniform. Other investigators[26,28,35] also obtained approximate solution for effective permeability similar to Eqs. (II-6) and (II-7).

It can be seen that if the magnitude of σ_Y is small, the effective permeabilities of one- and three-dimensional systems are close to the geometric mean. Warren and Price[3] used small values of σ_Y in their investigation. This led them to conclude that the geometric mean is a good approximation of effective permeability for three-dimensional systems.

5. Other related work

Other important areas of investigation are geostatistical approaches, conditional simulation, and statistical approaches to the inverse problem. Geostatistical approaches have been applied to groundwater problems by many investigators.[32,36-39] Delhomme[36] and Gambolati and Volpi[38,39] used geostatistical methods mainly for interpolation of transmissivity or hydraulic head. Delhomme[37] used conditional simulation to generate different two-dimensional transmissivity fields that all have the same spatial variability as the true field and are consistent with the measured transmissivity values at well locations. He investigated the effect of conditioning of log-transmissivity on the uncertainty of log-transmissivity and on the uncertainty of head values.

Statistical approaches to the inverse problem have also been one of the fields of intensive study in groundwater modeling (e.g., Refs. 40-47). In these approaches, both the parameter (permeability, conductivity, or transmissivity) and the pressure or head data were used to estimate parameter values at unsampled points in space.

B. Two-phase Flow Problems

For two-phase flow problems, Warren and Cosgrove[48] applied statistical methods to predict waterflood behaviour in a stratified system while Smith and Brown[49] used these methods on one-dimensional systems connected in series as well as on two-dimensional systems. Warren and Cosgrove[48] assumed that the permeability could be characterized by a log-normal distribution and the hydrocarbon pore volume by a normal distribution. From the results of their study they concluded that:

1. the effect of cross-flow in a statified system can be appreciable, particularly at very favorable or very unfavorable mobility ratios;
2. under normal conditions, the effect of variations in the hydrocarbon pore volume can be neglected; and
3. the failure to use all of the available permeability

data can lead to large errors in the prediction of the behaviour of a stratified reservoir.

Smith and Brown[49] studied problems of two-phase flow in porous media using Monte Carlo methods. They used the nearest-neighbor scheme, similar to the one used by Smith and Freeze,[20,21] to generate values of permeability and other related properties (e.g., porosity, relative permeability). Smith and Brown studied both one- and two-dimensional systems. They assumed that the viscosity ratio, μ_o/μ_w, was equal to unity and that capillary pressure could be neglected. Both steady-state and transient flows were investigated. They concluded that:

1. For steady-state and transient flows in one-dimensional systems, the differences between the pressure gradient from the set of Monte Carlo simulations and the harmonic average results are large, while for steady-state flow in two-dimensional systems, the differences are much smaller (in this case the geometric average results are used for comparison).
2. The pressure gradient is highly sensitive to the imposed property variability in one-dimensional systems both for steady-state and transient flows, while it is much less sensitive in two-dimensional systems with steady-state flow.
3. Two-dimensional numerical models are preferable to one-dimensional models for studying the effect of variation of porous medium properties on the resulting

pressure and saturation of two-phase flow systems.

III. STOCHASTIC METHODS

Before going into detail about stochastic methods some statistical terminology will be mentioned briefly. Because of the nature of the variables and parameters utilized in the flow equations, the discussion will concentrate on continuous random variables.

A. Distribution function and probability density function

The first statistical terminology to be introduced here is the distribution function (or cumulative distribution function). The distribution function, denoted by F(x), is defined as the probability that a random variable X assumes a value less than or equal to x or

$$F(x) = P(X \leq x), \qquad x \in R \qquad \text{(III-1)}$$

where $P(X \leq x)$ is the probability that X is less than or equal to x and R is the continuous domain ($-\infty$ to $+\infty$). In Eq. (III-1), X represents the random variable and x represents the real variable or realization which includes all of the possible values of the random variable.

Another useful function in probability theory is the probability density function. It is defined as the first derivative ~~of~~ the distribution function,

$$f(x) = \frac{dF(x)}{dx}, \qquad \text{(III-2)}$$

where f(x) is the probability density function which is defined only when the distribution function is differentiable. The distribution function and the probability density function are the complete definitions of a continuous random variable.

B. Mean, variance, covariance, and correlation function

The best known measure of central tendency is the expected value or the mean. It is defined by

$$\mu_x = E\{X\} = \int_{-\infty}^{\infty} x f(x) dx, \qquad \text{(III-3)}$$

for a continuous random variable, X. The operator E is the expectation operator. The common measure of the spread or dispersion of the possible values of a random variable around its mean value is the variance or its positive square root, the standard deviation. The variance is defined by

$$\mathrm{Var}\{X\} = \sigma_x^2 = E\{(X-\mu_x)^2\} = \int_{-\infty}^{\infty} (x-\mu_x)^2 f(x) dx. \qquad \text{(III-4)}$$

Where there is a relationship between input and output random variables, it is desirable to evaluate the probability density function of the output random variables provided that the probability density function of the input random variables are known. For example, let $Z = h(X_1, X_2, \ldots., X_n)$, where Z is the output random variable and $X_1, X_2, \ldots., X_n$ are input random variables. If h has a simple form, the technique called transformation of variables can be used and the probability density function of Z can be evaluated exactly in terms of the joint probablity density function of X_1, X_2, ..., X_n. However, in the situation involving complex systems, where the functional form of h is not known, approximate techniques, such as generation of system moments and Monte Carlo simulation must be used.[50] The generation of system moments method consists of expanding $h(X_1, X_2, ..., X_n)$ about $[E\{X_1\}, E\{X_2\}, ..., E\{X_n\}]$, the point at which each of the component variables takes on its expected value, by a multivariable Taylor series. Then, the moments of Z, e.g. mean, variance, can be obtained from manipulation of this Taylor series expression. Because of the complexity involved, the higher order terms in the Taylor series are usually neglected and the evaluated moments of Z are only approximate values.

When the concept of a random variable is expanded to describe a phenomenon occurring through time or in space a set of random variables or a random function is needed. At a particular point in space, the possible values of, say, permeability (or porosity, pressure, etc.) can be described by a random variable $X(\underline{s})$ and its associated probability distribution. The observed value or realization $x(\underline{s})$ at point $\underline{s}$ is then regarded as one of the infinity of values which the random variable $X(\underline{s})$ might have taken at point $\underline{s}$. The behaviour of permeability at all points in space can be described by a set of random variables $\{X(\underline{s})\}$ where the space variable $\underline{s}$ is a vector whose number of components is dependent on the dimension of the space considered. Thus, the statistical properties of permeability are described by associating probability distributions with a set of points $\underline{s}_1, \underline{s}_2, \ldots\ldots, \underline{s}_n$. The set of random variables $\{X(\underline{s})\}$ and its associated probability distributions are called a stochastic process.[51] (The set of random variables $\{X(\underline{s})\}$ is sometimes called a random function.[52]) The observed permeability value $x(\underline{s})$ is thus regarded as one of the doubly infinite set of functions which might have been generated by the stochastic process. The set is doubly infinite because an infinite set of values is possible at any given point and there are an infinite number of points in space. This interpretation of the stochastic process for permeability is analogous to the interpretation of time series by Jenkins and Watts.[51]

To describe a system of two random variables, it is useful to use, besides the expected values and variance of the components, certain other characteristics, two of which are the covariance and the correlation function. If the two random variables are from the same stochastic process, e.g. the process for permeability, the terms autocovariance and autocorrelation function are used. The autocovariance is defined by

$$C_{xx}(\underline{s}_1,\underline{s}_2) = E\{[X(\underline{s}_1)-E\{X(\underline{s}_1)\}][X(\underline{s}_2)-E\{X(\underline{s}_2)\}]\}, \tag{III-5}$$

and the autocorrelation function by

$$\rho_{XX}(\underline{s}_1,\underline{s}_2) = \frac{C_{XX}(\underline{s}_1,\underline{s}_2)}{\sigma_X(\underline{s}_1)\sigma_X(\underline{s}_2)}, \qquad \text{(III-6)}$$

where $C_{XX}(\underline{s}_1,\underline{s}_2)$ is the autocovariance function, $\rho_{XX}(\underline{s}_1,\underline{s}_2)$ is the autocorrelation function, and $\sigma_X(\underline{s}_1)$ and $\sigma_X(\underline{s}_2)$ are the variances of X at points $\underline{s}_1$ and $\underline{s}_2$.

If the two random variables are from different stochastic processes, e.g. the process for permeability and the process for porosity, the terms cross-covariance and cross-correlation function are used. The cross-covariance function is defined by

$$C_{XY}(\underline{s}_1,\underline{s}_2) = E\{[X(\underline{s}_1)-E\{X(\underline{s}_1)\}][Y(\underline{s}_2)-E\{Y(\underline{s}_2)\}]\}, \qquad \text{(III-7)}$$

and the cross-correlation function by

$$\rho_{XY}(\underline{s}_1,\underline{s}_2) = \frac{C_{XY}(\underline{s}_1,\underline{s}_2)}{\sigma_X(\underline{s}_1)\sigma_Y(\underline{s}_2)}, \qquad \text{(III-8)}$$

where $C_{XY}(\underline{s}_1,\underline{s}_2)$ is the cross-covariance function, $\rho_{XY}(\underline{s}_1,\underline{s}_2)$ is the cross-correlation function, and $\sigma_X(\underline{s}_1)$ and $\sigma_Y(\underline{s}_2)$ are the variances of X and Y at points $\underline{s}_1$ and $\underline{s}_2$, respectively.

It should be noted that the cross-covariance function is not symmetric. The cross-covariance function has the following properties:

$$C_{XY}(\underline{s}_1,\underline{s}_2) = C_{YX}(\underline{s}_2,\underline{s}_1),$$

and

$$C_{XY}(\underline{s}_1,\underline{s}_2) \neq C_{XY}(\underline{s}_2,\underline{s}_1).$$

(See Refs. 52 and 53.)

Another statistical characteristic which has been used extensively in geostatistics is the variogram. The variogram is defined as the variance of an increment $[X(\underline{s}_1)-X(\underline{s}_2)]$.[53,54] It is written as

$$2\gamma(\underline{s}_1,\underline{s}_2) = \mathrm{Var}\{X(\underline{s}_1)-X(\underline{s}_2)\}. \qquad \text{(III-9)}$$

C. Integral scale

The integral scale, λ_X, is defined by

$$\lambda_X = \int_0^\infty \rho_{XX}(h)dh, \qquad \text{(III-10)}$$

where h is a scalar quantity. The integral scale has been used in the theory of turbulence[2,55] and has served primarily as a guide for determining the time interval required to obtain stable (low variance) estimates of the (ensemble) mean of fluctuating quantities such as velocity and pressure. In particular, the ratio T/λ is interpreted as the equivalent number of independent observations contained in a sampling interval T. Vanmarcke[56] has called λ a scale of fluctuation and given a detailed discussion of its usefulness to characterize second-order properties and to interpret, estimate, and analyze patterns of random variation. More recently, the integral scale has been used in flow through porous media problems as one of the parameters that represent correlation of log-permeability in space.[17,18,26] It is interpreted as the average distance over which log-permeability is correlated.[17,18]

A functional form of $\rho_{xx}(h)$ that is consistent with Eq. (III-10) is

$$\rho_{xx}(h) = \exp(-h/\lambda_x). \qquad \text{(III-11)}$$

For a discrete one-dimensional system with blocks of equal size, Δs, Eq. (III-11) can be written as

$$\rho_k = \exp(-k\Delta s/\lambda_x) = [\exp(-\Delta s/\lambda_x)]^k = \rho^k, \qquad \text{(III-12)}$$

where ρ is an autocorrelation of lag 1 and ρ_k is an autocorrelation of lag k, k = 2, 3, ... Here ρ_k is written in the simple form of ρ where ρ is, in turn, a function of integral scale, λ_x, and block size, Δs. The relationship between ρ_k and ρ will be used later when the bias of sample variance and sample autocovariance of the generated values of a random variable (Chapter V) is determined. However, it should be noted that the relationship in Eq. (III-11) is for a continuous system whereas the relationship in Eq. (III-12) is for a discrete system.

Concerning the magnitude of integral scale, which has units of length, Gelhar[6,8] presented tables that show ranges of magnitude of integral scale. The magnitudes of integral scale vary from 0.1 to 1 m for vertical sampling and from 0.1 to 45000 m for horizontal sampling. These data are obtained from a wide range of porous media; that is, from soil to large aquifers. However, it should be recognized that the horizontal aquifer data usually represent a depth-averaged property, the transmissivity, whereas the horizontal soil data are smaller scale measurements.

Gelhar[8] pointed out that spatial correlation scales (or integral scales) should not be viewed in any absolute sense but rather will depend on the scale of the problem at hand. Another important point about scale is that the scale of heterogeneity is not an absolute quantity but rather it depends on the scale of the problem. Gelhar[8] stated that, for a stochastic approach to be applicable, the integral scale must be small relative to the problem scale.

D. Statistically homogeneous and statistically isotropic

The most important property of a random function (or stochastic process) which makes it possible to apply various methods of investigation is the dependence or independence of the properties of the random function on the origin of measurement (of time or space). In accordance with this, statistically homogeneous (stationary, in time series terminology) and statistically nonhomogeneous (non-stationary) random functions are distinguished. In the case of statistical homogeneity all the multi-dimensional density functions depend only on the relative positions of the points (in space) $\underline{s}_1$, $\underline{s}_2$, ..., $\underline{s}_n$, but not on the individual location of these quantities. In other words, for statistical homogeneity of a random function, the relation

$$f_{X1,X2,\ldots\ldots,Xn}(\underline{s}_1,\underline{s}_2,\ldots\ldots,\underline{s}_n) = f_{X1,X2,\ldots\ldots,Xn}(\underline{s}_1+\underline{h},\underline{s}_2+\underline{h},\ldots\ldots,\underline{s}_n+\underline{h}), \tag{III-13}$$

must be satisfied for any n where $\underline{h}$ is any vector. [Here the subscripts $X1 = X(\underline{s}_1)$, ..., $Xn = X(\underline{s}_n)$.] This is called statistical homogeneity of order n. It should be noted that a random function which is statistically homogeneous of order n is also statistically homogeneous of order $k < n$, since the lower-order density function may be determined from the nth-order density function.[57] If a random function is in a strict sense statistically homogeneous, it is statistically homogeneous of order n for all $n = 1, 2, \ldots\ldots$

A random function is said to be statistically homogeneous in the wide-sense (second-order statistically homogeneous) if its expected value is a constant and its autocovariance function depends only on $\underline{s}_1 - \underline{s}_2 = \underline{h}$:

$$\mu_X = E\{X(\underline{s})\} = \text{constant}, \qquad \text{(III-14)}$$

and

$$C_{XX}(\underline{h}) = E\{[X(\underline{s}+\underline{h})-\mu_X][X(\underline{s})-\mu_X]\}. \qquad \text{(III-15)}$$

Therefore, for second order statistical homogeneity, Eqs. (III-5) to (III-8) become

$$C_{XX}(\underline{h}) = E\{[X(\underline{s}+\underline{h})-E\{X(\underline{s}+\underline{h})\}][X(\underline{s})-E\{X(\underline{s})\}]\}, \text{(III-16)}$$

$$\rho_{XX}(\underline{h}) = \frac{C_{XX}(\underline{h})}{\sigma_X^2}, \qquad \text{(III-17)}$$

and

$$C_{XY}(\underline{h}) = E\{[X(\underline{s}+\underline{h})-E\{X(\underline{s}+\underline{h})\}][Y(\underline{s})-E\{Y(\underline{s})\}]\}, \text{(III-18)}$$

$$\rho_{XY}(\underline{h}) = \frac{C_{XY}(\underline{h})}{\sigma_X\sigma_Y}. \qquad \text{(III-19)}$$

In addition to statistical homogeneity, if a random function is statistically isotropic, Eqs. (III-16) and (III-17) become

$$C_{XX}(h) = E\{[X(\underline{s}+h)-E\{X(\underline{s}+h)\}][X(\underline{s})-E\{X(\underline{s})\}]\}, \text{(III-20)}$$

and

$$\rho_{XX}(h) = \frac{C_{XX}(h)}{\sigma_X^2}. \qquad \text{(III-21)}$$

Similarly, Eqs. (III-18) and (III-19) become

$$C_{XY}(h) = E\{[X(\underline{s}+h)-E\{X(\underline{s}+h)\}][Y(\underline{s})-E\{Y(\underline{s})\}]\}, \text{(III-22)}$$

and

$$\rho_{XY}(h) = \frac{C_{XY}(h)}{\sigma_X\sigma_Y}, \qquad \text{(III-23)}$$

where h is a scalar quantity.

On the other hand, if $[X(\underline{s}+\underline{h})-X(\underline{s})]$ is stationary or statistically homogeneous, $X(\underline{h})$ is said to be a process with stationary increments,[58] or intrinsic, using geostatistical terminology. For a process with stationary increments, Eq. (III-9) can be written as

$$2\gamma(\underline{h}) = \mathrm{Var}\{X(\underline{s}+\underline{h})-X(\underline{s})\}. \qquad \text{(III-24)}$$

Stationarity in increments is less restrictive than second-order stationarity, i.e. second-order stationarity implies stationarity in increments but the converse is not true. Gutjahr and Gelhar[24] and Dagan[29] are among the authors who apply the concept of stationarity in increments (or the intrinsic hypothesis) in groundwater flow problems.

E. Stochastic methods

The stochastic methods used to solve a stochastic differential equation may be classified into (1) full distribution techniques and (2) first- and second-moment analyses. The full distribution techniques generally used are Monte Carlo simulations. These methods were used by Warren and Price[3] to investigate the effective permeability of the unidirectional flow in three-dimensional systems for both steady and transient flows. Freeze,[4] later, used the Monte Carlo simulation method to determine steady and transient flows in a one-dimensional system. Smith and Freeze[20,21] expanded Freeze's work[4] by investigating steady groundwater flow in bounded domains of one- and two-dimensional systems using the Monte Carlo simulation method with a first-order, nearest-neighbor autocorrelation model. Smith and Brown[49] also used the Monte Carlo simulation method to solve waterflooding problems.

The first- and second-moment methods that are commonly used are perturbation and Taylor series expansion. In perturbation analysis, the partial differential equation governing the flow phenomena is perturbed slightly, yielding a new equation which governs the random or fluctuating component of the variable of interest. The perturbation method using a spectral representation technique has been employed, for example, by Gelhar and his coworkers.[15,17,18,22-25] Tang and Pinder[59] used perturbation in space to solve groundwater flow and mass transport problems. Analyses based on Taylor series expansions, on the other hand, generally expand analytical or numerical solutions of a flow-governing equation around the expected values of the solution's parameters and independent variables. These series expansions may then be used to deduce the probabilistic moments of the variable of interest (see Ref. 60). The main restriction of the first- and second-moment methods is that for the perturbation method the perturbation terms have to be small compared to their means whereas for the Taylor series expansion method the higher order terms must be small compared to the term (or terms) retained in the Taylor series. These conditions imply that the variance of the input random variables must be small compared to their mean. Monte Carlo simulations do not have this restriction. However, when the variances of the input random variables are large, numerical difficulties may become more serious.[21] The main virtue of Monte Carlo simulations is their generality. However, this generality is achieved at the expense of time-consuming calculations on a computer, and only numerical results for a particular case are obtained.

Except for the perturbation method using spectral theory, all the methods mentioned have been carried out numerically. The spectral techniques provide a continuous description of the porous medium properties. This continuous description is certainly the most realistic and is the main advantage of the spectral techniques over other methods.

Another advantage of the spectral techniques is the analytical nature of the solutions.

In addition to the methods mentioned, Dagan[26,27] has used the method of embedding matrix and the self-consistent approximation concept (see Refs. 34 and 61) to solve the stochastic differential equations.

Another approach to solving the flow through porous medium problem is the geostatistical approach. This approach concentrates mainly on the problem of stochastic interpolation, i.e. the prediction of the best estimate and variances of estimation, on a spatial grid, of variables measured at a few points.[32,36-39]

IV. ASSUMPTIONS AND MATHEMATICAL FORMULATION

This chapter gives a detailed discussion of the systems to be studied in this work. The general assumptions used will be stated first while additional assumptions for specific derivations will be given when they are needed. The detailed derivation of the solutions to the flow equations using spectral techniques will be presented.

A. Scale

The question of scale concerning heterogeneity of porous media has been discussed by many authors.[62-66] In this study all calculations will start from the macroscopic scale. At the macroscopic scale, the Darcian continuum is applicable. The size of this macroscopic scale is equivalent to the representative elementary volume (REV) defined by Bear.[63] This volume must be larger than the size of a single pore. In fact, it must include a sufficient number of pores to permit the meaningful statistical average required in the continuum concept. For each REV all rock properties, such as permeability and porosity, can be represented by single values. However, it should be noted that the sizes of the REV for permeability and porosity can, in fact, be different. This problem will not be dealt with here and it will be assumed that all rock properties have the same REV size.

Clearly, a REV is relatively small compared to the scale of interest for flow in the reservoir. Therefore, the spectral techniques seem to be superior to the Monte Carlo techniques. However, both techniques are selected as tools to investigate the effect of heterogeneity in two-phase flow through a porous medium system. The spectral techniques are selected because of the analytical nature of the solution and because of the representation of spatial variations in a continuous sense. However, the spectral techniques have many limitations, e.g. all random variables must possess (second-order) statistically homogeneous properties and the system considered must be unbounded, usually. These shortcomings can be overcome by using the Monte Carlo techniques. Therefore, the Monte Carlo techniques are selected to study the bounded systems.

It should be noted that the REV is also smaller than the scale at which Darcy's equation is conventionally applied in petroleum engineering. This means that in the present-day application of Darcy's equation some kind of averaging is always implicitly implied. One of the purposes of this study is to seek an answer to the question: is it allowable to use one representative value of, say, permeability (to each phase) for a simulation cell in two-phase flow problems? As mentioned before in the literature survey section, the answer is affirmative for one-phase flow problems.

Many investigators[67-73] have used statistical techniques on problems at the microscopic scale. It should be emphasized that this study is not at all in this tradition.

B. Assumptions concerning the flow condition

In this study a simple flow condition will be used. The following assumptions are made:

1. The flow is incompressible, immiscible, and isothermal.
2. All fluids and porous medium properties do not depend upon pressure and temperature.
3. The flow is one-dimensional and is in the horizontal direction.

Because the assumptions concerning the probabilistic properties of permeability and porosity and also for relative permeabilities are different for different techniques of solution, these assumptions will be stated when needed.

C. Probability distributions of permeability and porosity

Freeze[4] reviewed the literature[74-79] and concluded that permeability (closely related to conductivity) is likely to have a log-normal distribution. In addition, Warren and Skiba,[9] using a statistical formulation, have provided a theoretical basis for this conclusion. Consequently, it will be assumed in this study that permeability is log-normally distributed. If the permeability is log-normally distributed, the parameter, $Y = \ln(k)$, will be normally distributed. This will be written as $Y : N[\mu_Y, \sigma_Y]$ where μ_Y is the mean and σ_Y the standard deviation. From the table presented by Freeze,[4] for sandstone, most values of σ_Z, where $Z = \log_{10}(k)$, are lower than 0.5. This is equivalent to a value of σ_Y of 1.15. In this study a σ_Y range of 0 to 0.8 will be considered.

It is usually assumed that porosity is approximately normally distributed, $\phi : N[\mu_\phi, \sigma_\phi]$ (see Refs. 4,9,48). New findings[80-82] suggest that, for shaly sandstone, there is a strong linear relationship between log-permeability and total porosity, which confirms these assumptions.

D. Exponential autocorrelation for log-permeability

Many functional forms for the autocorrelation function for log-permeability have been proposed (e.g., Refs 18,83). The exponential function is one of the functional forms that has been used. In general, there are many arguments against using the exponential autocorrelation function (for any random variables) for a system of more than one dimension (see Refs. 84-86). This point will be discussed in more detail later. In the following it will be shown that utilizing the exponential autocorrelation function for log-permeability is appropriate for the system studied here.

Any stochastic process in one-dimensional space may be represented by the one-dimensional nearest-neighbor scheme[84,85,87] as follows:

$$X_i = \beta[X_{i-1} + X_{i+1}] + \eta_i, \qquad \text{(IV-1)}$$

where X_i is a random variable or preferably a random function (or a stochastic process), β is a parameter, and η_i is an independent normal random variable with zero mean and variance, σ_η^2. The simple, though not rigorous, interpretation of Eq. (IV-1) is that the value of the random variable X at point i is dependent upon values of itself at points i-1 and i+1 and the independent normal random variable (white noise) at point i. The degree of dependence of X_i on its neighbors X_{i-1} and X_{i+1} is expressed through the magnitude of β. In Eq. (IV-1) the degree of dependence of X_i on X_{i-1} and X_{i+1} is the same. This implies that lateral symmetry has been assumed implicitly.

Subtracting $2\beta X_i$ from both sides of Eq. (IV-1) and reordering the terms yields the second-order difference equation:

$$\beta[X_{i-1} - 2X_i + X_{i+1}] - (1-2\beta)X_i = -\eta_i. \qquad \text{(IV-2)}$$

The associated second-order differential equation is[56,84]

$$\frac{d^2X(s)}{ds^2} - \alpha^2 X(s) = \eta(s), \qquad \text{(IV-3)}$$

where $\alpha^2 = (1-2\beta)/\beta$ in the limit when the distance goes to zero. Only the continuous-space analysis is pursued here.

For the case of statistically isotropic p-dimensional space with coordinates s_j, $j = 1, \ldots\ldots, p$, Eq. (IV-3) becomes[85]

$$(\nabla^2 - \alpha^2) X(\underline{s}) = \eta(\underline{s}), \tag{IV-4}$$

where $\underline{s}$ represents the location in p-dimensional space (the under-bar sign indicates a vector) and

$$\nabla^2 = \sum_{j=1}^{p} \frac{\partial^2}{\partial s_j^{\,2}}. \tag{IV-5}$$

Whittle[85] has shown that the spectral density function, $S_{X,p}(\underline{\omega})$, for a p-dimensional situation based on Eq. (IV-3) is

$$S_{X,p}(\underline{\omega}) = \frac{1}{(\omega^2 + \alpha^2)^2}, \tag{IV-6}$$

and the covariance function is

$$C_{X,p}(\underline{h}) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} e^{i\underline{\omega}\cdot\underline{h}}\, S_{X,p}(\underline{\omega})\, d\underline{\omega}$$

$$= \frac{h^{1-p/2}}{(2\pi)^{p/2}} \int_0^{\infty} \frac{J_{p/2-1}(\omega s)\, \omega^{p/2}\, d\omega}{(\omega^2 + \alpha^2)^2}$$

$$= \frac{(h/\alpha)^{2-p/2}\, K_{2-p/2}(\alpha h)}{2\Gamma(2)},$$

or

$$C_{x,p}(h) = \frac{1}{2} \left(\frac{h}{\alpha}\right)^{2-p/2} K_{2-p/2}(\alpha h), \qquad \text{(IV-7)}$$

where $J_r(t)$ is the Bessel function of order r, $K_r(t)$ is the modified Bessel function of the second kind of order r, $\Gamma(t)$ is the gamma function, $\underline{h}$ is a vector between any two points in the s-coordinate, h is the magnitude of $\underline{h}$ and is a positive quantity, and p is the number of dimensions. To obtain the last expression of Eq. (IV-7), it is assumed that the processes are statistically isotropic as well as stationary (i.e., statistically invariant under rotations as well as under translations).[85] The autocorrelation function can be written as

$$\rho_{x,p}(h) = \frac{C_{x,p}(h)}{C_{x,p}(0)}. \qquad \text{(IV-8)}$$

It follows that (Appendix B, see also Agterberg[88]),

$$p=1; \qquad \rho_{x,1}(h) = (1+\alpha h)\exp(-\alpha h), \qquad \text{(IV-9)}$$

$$p=2; \qquad \rho_{x,2}(h) = \alpha h K_1(\alpha h), \qquad \text{(IV-10)}$$

$$p=3; \qquad \rho_{x,3}(h) = \exp(-\alpha h). \qquad \text{(IV-11)}$$

Therefore, for statistically isotropic and statistically homogeneous three-dimensional systems, the autocovariance of a random variable has an exponential form, Eq. (IV-11). If the autocorrelation in three-dimensional space of a statistically isotropic and statistically homogeneous system is described by the exponential function, the autocorrelation function in one- or two-dimensional space will also be an exponential.[52,56] Then, it is permissible to use the exponential autocorrelation function for one- and two- dimensional space of a statistically isotropic and statistically homogeneous system whose complete dimensions are three.

To clarify this point let us consider the following. The discrete process that is equivalent to the continuous process which gives the correlation function form in Eq. (IV-9), p = 1, is [Eq. (IV-1) rewritten here]

$$X_i = \beta[X_{i-1}+X_{i+1}]+\eta_i, \qquad \text{(IV-1)}$$

It can be seen that in this process, Eq. (IV-1), only the correlation in the i-direction is accounted for. There is no correlation of X in the other directions. This process is useful for problems which have a complete dimension of one, such as traffic flow on a road which has no (or widely spaced) intersections. Therefore, the correlation form in Eq. (IV-9) should be used for problems which are one-dimensional in nature.

The continuous process which yields the correlation form in Eq. (IV-10), p = 2, is equivalent to the following discrete process,

$$X_{i,j} = \beta[X_{i-1,j}+X_{i+1,j}+X_{i,j-1}+X_{i,j+1}]+\eta_{ij}. \qquad \text{(IV-12)}$$

This process takes into account the correlation in the i- and j-directions. It is useful for problems whose complete dimensions are two, such as growth of trees on a plane. Then, the autocorrelation function given in Eq. (IV-10) should be used for problems which are two-dimensional in nature.

For the autocorrelation function in Eq. (IV-11), the equivalent discrete process is

$$X_{i,j,k} = \beta[X_{i-1,j,k}+X_{i+1,j,k}+X_{i,j-1,k}+X_{i,j+1,k}+X_{i,j,k-1}+X_{i,j,k+1}]+\eta_{ijk}. \qquad \text{(IV-13)}$$

It is clear that this process should be used for problems that have a complete dimension of three, such as deposition of sediments in three-dimensional space. Therefore, the autocorrelation function form given in Eq. (IV-11) should be used for problems that have a correlation in three dimensions.

For the reasons stated above, it can be argued that for problems of two-phase flow through a one-dimensional porous medium, such as those considered in this study, the exponential autocorrelation function given in Eq. (IV-11) should be used. This is because, generally, the deposition of sediments takes place in three-dimensional space, and permeabilty or log-permeability should be regarded as having a correlation in all three dimensions. For the reasons mentioned one should be, then, cautious when using the nearest-neighbor scheme to generate values or realizations of log-permeability or other random variables. That is, generally, log-permeability or other rock properties are correlated in three dimensions due to the nature of deposition of sediments. Consequently, one should use the scheme presented in Eq. (IV-13), which includes correlation in three dimensions, to generate values of log-permeability, although the flow being considered is one- or two-dimensional in nature. It seems that Smith[19], Smith and Freeze[20,21] and Smith and Brown[49] were unaware of this point. Furthermore, for some properties, such as transmissivity, which are properties averaged over the thickness, the nearest-neighbor scheme given in Eq. (IV-12) seems to be the most appropriate one.

Another interesting point about the nearest-neighbor schemes, Eqs. (IV-1), (IV-12), (IV-13), is that these processes will be statistically homogeneous when $2|\beta| < 1$ in Eq. (IV-1), $4|\beta| < 1$ in Eq. (IV-12), and $6|\beta| < 1$ in Eq. (IV-13).[84,89] This restriction should be kept in mind when one wants to generate values for statistically homogeneous random variables. Failing to recognize this limitation may lead to wrong conclusions about the effect of statistical homogeneity of an input random variable on the behavior of an output random variable. In fact, some of the runs by Smith[19] represent cases of statistical nonhomogeneity in log-conductivity. Therefore, when comparing the results from other methods which assume statistical homogeneity in log-conductivity, e.g. spectral techniques, with Smith's work, caution should be exercised in order to use only the results that represent cases of statistical homogeneity.

Let us go back to the objection of using an exponential autocorrelation function in a two-dimensional problem, though this is not of direct concern to this study. In the case of two-dimensional systems, Matern[90] showed that the exponential autocorrelation function corresponds to the spectral density function $(\omega_1^2 + \omega_2^2 + \alpha^2)^{-3/2}$. Whittle[84] determined the simplest process which corresponds to this form of the spectral density function. This process is

$$[(\frac{\partial}{\partial s_1})^2 + (\frac{\partial}{\partial s_2})^2 - \alpha^2]^{3/4} X(s_1,s_2) = \eta(s_1,s_2). \quad \text{(IV-14)}$$

He suggested that it is difficult to visualize a physical mechanism which would lead to the relation defined by Eq. (IV-14) and has rejected the use of an exponential autocorrelation function for two-dimensional problems. This is logical because it is indeed difficult to give a meaningful physical interpretation for the process in Eq.

(IV-14). However, this problem arises only in the case of processes whose complete dimensions are two, but not in the case of processes whose complete dimensions are three, such as the ones in this study. It has been shown that using Eq. (IV-13) will lead to an autocorrelation function which has an exponential form (see Appendix B). It should be added here that the objection to using the exponential autocorrelation function for a random variable in space has also been discussed in Bartlett's work.[86,87]

Before leaving this topic an example of the autocorrelation function for log-permeability is shown in Figure IV-1. In this figure the symbols represent data points and the solid curve represents the exponential fit to the data. It can be seen that the exponential function fits the data quite well.

E. Spectral techniques

Spectral techniques have been used extensively by Gelhar and his coworkers[17,18,23-25] to study one-phase flow through a heterogeneous system. The following derivation for steady-state two-phase flow through a porous medium will follow their work. In order to utilize spectral techniques, the following assumptions are made:

1. The flow is steady-state,
2. Capillary pressure can be neglected,
3. The relative permeabilities to oil and water are the same for all parts of the system and they are dependent

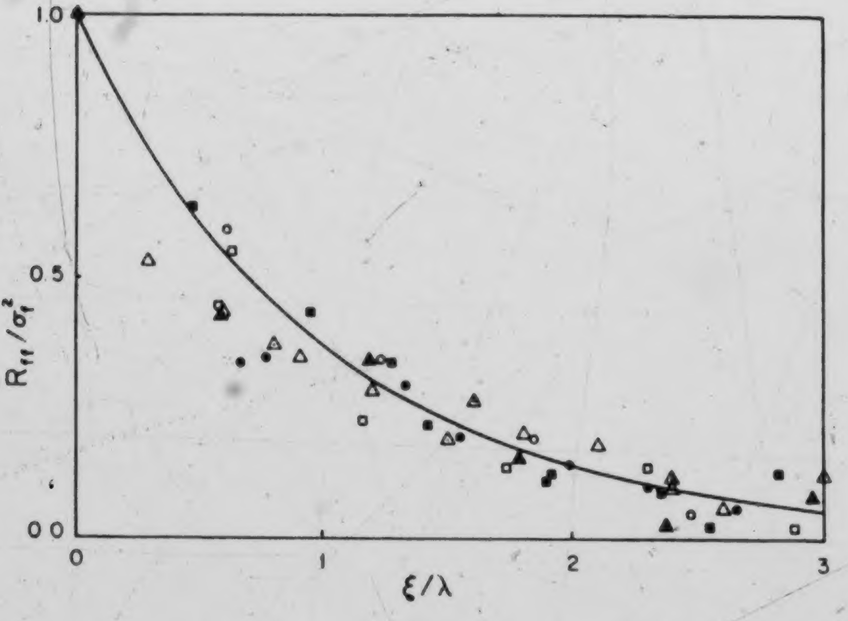


Figure IV-1 Autocorrelation function for ln(k) from 8 boreholes in the Mt. Simon aquifer compared with exponential function, the solid line (from Bakr, 1976; see Ref. 6).

only upon water saturation,

4. Porosity is a deterministic parameter,
5. The mean or expected value of log-permeability is constant throughout the system,
6. Both log-permeability and pressure are statistically homogeneous, and log-permeability is also assumed to be statistically isotropic,
7. The system is unbounded.

Assumptions 2 to 4 are rather unrealistic. They are made so that the analysis by spectral techniques is tractable.

For the sake of simplicity, the derivation for a one-dimensional system will be presented. For further information about spectral techniques the books by Cox and Miller,[91] Sveshnikov,[52] and Jenkins and Watts[51] are recommended.

Let us start with Darcy's equations for each phase,

$$q_o = -\frac{kk_{ro}A}{\mu_o}\frac{\partial p_o}{\partial x}, \qquad \text{(IV-15)}$$

and

$$q_w = -\frac{kk_{rw}A}{\mu_w}\frac{\partial p_w}{\partial x}. \qquad \text{(IV-16)}$$

The continuity equations are

$$\frac{\partial q_o}{\partial x} = -\phi A\frac{\partial S_o}{\partial t}, \qquad \text{(IV-17)}$$

and

$$\frac{\partial q_w}{\partial x} = -\phi A \frac{\partial S_w}{\partial t}. \qquad \text{(IV-18)}$$

Combining Eqs. (IV-17) and (IV-18) yields

$$\frac{\partial q_o}{\partial x} + \frac{\partial q_w}{\partial x} = 0, \qquad \text{(IV-19)}$$

because $S_w+S_o = 1$. Substituting Eqs. (IV-15) and (IV-16) into Eq. (IV-19) and rearranging, Eq. (IV-19) becomes

$$\frac{\partial}{\partial x}\left[kA\left(\frac{k_{ro}}{\mu_o}\frac{\partial p_o}{\partial x} + \frac{k_{rw}}{\mu_w}\frac{\partial p_w}{\partial x}\right)\right] = 0. \qquad \text{(IV-20)}$$

Eq. (IV-20) is valid for both steady-state and transient two-phase flow. Let us consider only steady-state flow and let

$$P_c = p_o - p_w,$$

then,

$$\frac{\partial p_o}{\partial x} = \frac{\partial p_w}{\partial x} + \frac{\partial P_c}{\partial x}. \qquad \text{(IV-21)}$$

With the assumption that capillary pressure can be neglected, Eq. (IV-21) becomes

$$\frac{\partial p_o}{\partial x} = \frac{\partial p_w}{\partial x}. \qquad \text{(IV-22)}$$

Then, Eq. (IV-20) can be written as

$$\frac{\partial}{\partial x}\left[kA\left(\frac{k_{ro}}{\mu_o} + \frac{k_{rw}}{\mu_w}\right)\frac{\partial p_o}{\partial x}\right] = 0. \qquad \text{(IV-23)}$$

If A is constant along the system and k_{rw} and k_{ro} are functions of S_w only, Eq. (IV-23) becomes

$$A\left[\frac{k_{ro}}{\mu_o} + \frac{k_{rw}}{\mu_w}\right]\frac{\partial}{\partial x}\left[k\frac{\partial p_o}{\partial x}\right] = 0,$$

or

$$\frac{\partial}{\partial x}\left[k\frac{\partial p_o}{\partial x}\right] = 0. \qquad \text{(IV-24)}$$

The form of Eq. (IV-24) is similar to the form for one-phase flow and can be solved as follows. Let

$$p_o = P+p, \; P = E\{p_o\}, \; E\{p\} = 0, \qquad \text{(IV-25)}$$

and

$$\ln(k) = F+f, \; F = E\{\ln(k)\} \equiv \ln(k_G), \; E\{f\} = 0, \qquad \text{(IV-26)}$$

where P and F are means and p and f are the random parts of p_o and ln(k), respectively, and E is an expectation operator. Substituting Eqs. (IV-25) and (IV-26) into Eq. (IV-24),

$$\frac{\partial}{\partial x}\left[e^{F+f}\frac{\partial}{\partial x}(P+p)\right] = 0,$$

or

$$e^{F+f}\frac{\partial^2}{\partial x^2}(P+p)+\frac{\partial}{\partial x}(P+p)e^{F+f}\left[\frac{\partial F}{\partial x}+\frac{\partial f}{\partial x}\right] = 0,$$

or

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 p}{\partial x^2} + \frac{\partial P}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial P}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial p}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial p}{\partial x}\frac{\partial f}{\partial x} = 0. \qquad \text{(IV-27)}$$

Neglecting the products of perturbations and with the assumption that F is not a function of location, Eq. (IV-27) becomes

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 p}{\partial x^2} + \frac{\partial P}{\partial x}\frac{\partial f}{\partial x} = 0. \qquad \text{(IV-28)}$$

Taking the expectation of Eq. (IV-28),

$$\frac{\partial^2 P}{\partial x^2} = 0, \qquad \text{(IV-29)}$$

which is the one-dimensional Laplace equation. This means that the mean oil pressure (and also the mean water pressure) is a linear function of distance, x. Subtracting Eq. (IV-29) from Eq. (IV-28) produces the following equation for the perturbations:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial P}{\partial x}\frac{\partial f}{\partial x} = 0. \qquad \text{(IV-30)}$$

The mean oil pressure gradient can be evaluated by using Eq. (IV-15),

$$k\frac{\partial p_o}{\partial x} = -\frac{q_o\mu_o}{k_{ro}A}. \qquad \text{(IV-31)}$$

Substituting for k and p_o and rearranging, Eq. (IV-31) becomes

$$\frac{\partial P}{\partial x} + \frac{\partial p}{\partial x} + f\frac{\partial P}{\partial x} + f\frac{\partial p}{\partial x} = -\frac{q_o\mu_o}{k_{ro}k_G A}. \qquad \text{(IV-32)}$$

where $k_G = e^F$ and e^f has been approximated by 1+f. Taking the expectation, and neglecting the product of the perturbations, Eq. (IV-32) becomes

$$\frac{\partial P}{\partial x} = -\frac{q_o\mu_o}{k_{ro}k_G A} = -J, \qquad \text{(IV-33)}$$

where q_o is assumed to be a deterministic quantity. It should be noted that J in Eq. (IV-33) is a constant for a specified condition (constant saturation) of steady-state flow. Although only k_{ro} and μ_o appear in Eq. (IV-33), J is influenced also by k_{rw} and μ_w through the value of q_o. Substituting Eq. (IV-33) into Eq. (IV-30),

$$\frac{\partial^2 p}{\partial x^2} - J\frac{\partial f}{\partial x} = 0. \qquad \text{(IV-34)}$$

In solving Eq. (IV-34) for the covariance function of the oil pressure in terms of the covariance function of log-permeability, it is assumed that p and f can be written in Fourier-Stieltjes integral form as

$$p(x) = \int_{-\infty}^{\infty} e^{ix\omega} dZ_p(\omega), \qquad \text{(IV-35)}$$

and

$$f(x) = \int_{-\infty}^{\infty} e^{ix\omega} dZ_f(\omega), \qquad \text{(IV-36)}$$

where $Z_p(\omega)$ and $Z_f(\omega)$ are the spectral functions. The functions $Z_p(\omega)$ and $Z_f(\omega)$ are non-decreasing bounded functions of their arguments. In fact, it can be shown that any stationary processes, such as p(x) and f(x), will have the spectral representations as expressed in Eqs. (IV-35) and (IV-36).[52,91,92] Substituting Eqs. (IV-35) and (IV-36) into Eq. (IV-34),

$$\int_{-\infty}^{\infty} \omega^2 e^{ix\omega} dZ_p(\omega) = -J\int_{-\infty}^{\infty} i\omega e^{ix\omega} dZ_f(\omega). \qquad \text{(IV-37)}$$

Multiplying both sides of Eq. (IV-37) by their conjugates, rearranging, and taking the expectation,

$$\iint_{-\infty}^{\infty} \omega^2 \omega'^2 e^{i(x\omega - x'\omega')} E\{dZ_p(\omega) dZ_p^*(\omega')\}$$
$$= J^2 \iint_{-\infty}^{\infty} \omega\omega' e^{i(x\omega - x'\omega')} E\{dZ_f(\omega) dZ_f^*(\omega')\}, \qquad \text{(IV-38)}$$

where ω', x', Z_p^* and Z_f^* are the conjugates of ω, x, Z_p and Z_f, respectively. For the expectation terms in Eq. (IV-38), let us consider the following. Multiplying Eq. (IV-36) by its conjugate and taking the expectation,

$$C_{ff}(x,x') = E\{f(x)f^*(x')\} = \iint_{-\infty}^{\infty} e^{i(x\omega - x'\omega')} E\{dZ_f(\omega)dZ_f^*(\omega')\}, \tag{IV-39}$$

where $C_{ff}(x,x')$ is an autocovariance function of f. If it is assumed that the process f(x) is second-order statistically homogeneous (or second-order stationary), the integral in Eq. (IV-39) must depend only on the difference $h = x-x'$. This is the case only when the product of differentials $dZ_f(\omega)dZ_f^*(\omega')$ satisfies the following relation (see Ref. 52),

$$E\{dZ_f(\omega)dZ_f^*(\omega')\} = S_{ff}(\omega)\delta(\omega-\omega')d\omega d\omega', \tag{IV-40}$$

where $S_{ff}(\omega)$ is called the spectral density function of f and $\delta(\omega-\omega')$ is the Dirac delta function. Substituting Eq. (IV-40) into Eq. (IV-39) and carrying out the algebra,

$$C_{ff}(h) = \int_{-\infty}^{\infty} e^{i\omega h} S_{ff}(\omega)d\omega, \tag{IV-41}$$

because

$$\int_{-\infty}^{\infty} g(\omega')\delta(\omega-\omega')d\omega' = g(\omega).$$

Eq. (IV-41) expresses the relationship between the autocovariance function and the spectral density function of f which is second-order statistically homogeneous. Because p is also assumed to be second-order statistically homogeneous, a similar relationship can be written

$$E\{dZ_p(\omega)dZ_p^*(\omega')\} = S_{pp}(\omega)\delta(\omega-\omega')d\omega d\omega', \qquad \text{(IV-42)}$$

and

$$C_{pp}(h) = \int_{-\infty}^{\infty} e^{i\omega h} S_{pp}(\omega)d\omega, \qquad \text{(IV-43)}$$

where $C_{pp}(h)$ is the covariance function and $S_{pp}(\omega)$ is the spectral density function of p. The inverse transforms of Eqs. (IV-41) and (IV-43) can be written as

$$S_{ff}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega h} C_{ff}(h)dh, \qquad \text{(IV-44)}$$

and

$$S_{pp}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega h} C_{pp}(h)dh, \qquad \text{(IV-45)}$$

respectively. Substituting Eqs. (IV-40) and (IV-42) in Eq. (IV-38) and manipulating, Eq. (IV-38) becomes

$$S_{pp}(\omega) = \frac{J^2}{\omega^2} S_{ff}(\omega), \qquad \text{(IV-46)}$$

which is the relation between spectral density functions of p and f for a one-dimensional system.

For two-dimensional and three-dimensional systems, the relation between the spectral density functions of p and f can be written as

$$S_{pp}(\underline{\omega}) = \frac{J_1^2\omega_1^2}{\omega^4} S_{ff}(\underline{\omega}), \qquad \text{(IV-47)}$$

where

$$J_1 = \frac{q_{o1}\mu_o}{k_{ro}k_G A},$$

and q_{o1} is the oil flow rate in the main flow direction, $\underline{\omega} = (\omega_1, \omega_2)$ for a two-dimensional system, $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ for a three-dimensional system, and ω is the magnitude of $\underline{\omega}$.

It can be seen that the relationship between spectral density functions of p and f for one-, two-, and three-dimensional systems of two-phase flows are the same as the ones for one-phase flows except that J or J_1 for two-phase flows also depends on relative permeabilities and viscosities of the two fluids. Therefore, all conclusions about the variance and autocovariance of pressure from one-phase flow studies can be applied to this simplified steady-state two-phase flow system, provided the proper definition of J is used.

In fact, the expression for J can be written as

$$J = \frac{(q_o/q)q\mu_o}{k_{ro}k_G A} = \frac{(1-f_w)q\mu_o}{k_{ro}k_G A}, \qquad \text{(IV-48)}$$

where $f_w = q_w/q$, and q is the total flow rate. Then for a specific pair of fluids for the two-phase flow system, the variance and autocovariance of the oil pressure will change with saturation. This is because f_w and k_{ro} are functions of saturation.

Using the exponential autocorrelation function for f or log-permeability, an attempt has been made to obtain expressions for the autocovariance of oil pressure for one-, two- and three-dimensional systems from Eqs. (IV-46) and (IV-47). It turned out that expressions for the statistically homogeneous and isotropic autocovariance function of oil pressure for all three systems cannot be obtained because of singularity problems. However, the attempt was successful in obtaining an expression for the variance of oil pressure for a three-dimensional system (see Appendix C). The expression for the three-dimensional system is

$$\sigma_p^2 = C_{pp}(0) = \frac{1}{3} J_1^2 \sigma_f^2 \lambda_f^2, \qquad \text{(IV-49)}$$

where λ_f is an integral scale of log-permeability. Expressions for the variance of oil pressure for one- and two-dimensional systems were not obtained because of a singularity problem.

From Eqs. (IV-49), because J_1 is dependent on saturation, the variance of oil pressure is also dependent upon saturation in addition to variance and integral scale of log-permeability and the other terms that define J_1.

V. MONTE CARLO SIMULATION TECHNIQUES

Hammersley and Handcomb[93] defined Monte Carlo methods as the methods which "comprise that branch of experimental mathematics which is concerned with experiments on random numbers." There are many applications of Monte Carlo methods in various fields. The particular technique used in this study is called a direct simulation by Hammersley and Handcombs.[93] This direct Monte Carlo simulation is used to solve the stochastic, two-phase flow problem by repetitively solving a set of deterministic flow problems, each of which is an equally probable representation of the response of the real heterogeneous medium.

The porous medium is divided into a finite number of equal-size blocks. Each block is assumed to have uniform rock properties. A set of values of permeability and porosity is assigned to the blocks. This is called a realization. The two-phase flow equations are used to solve for pressure and saturation for all blocks for specified initial and boundary conditions. Then, another set of values of permeability and porosity is assigned to the blocks. This is another realization. Again, the two-phase flow equations are utilized.

After repeating this procedure many times, say more than 200 times, sets of values of pressure and saturation for each block are obtained. These values of pressure and saturation are, then, used to estimate the means and variances of pressure and saturation for each block. These estimated means and variances of pressure and saturation correspond to the statistical properties of the input permeability and porosity.

In this study it is assumed that permeability is log-normally distributed and porosity is normally distributed. The spatial dependence between neighboring values of permeability or porosity is taken into account through a specified covariance matrix. This covariance matrix is, in turn, obtained through a specified variance, integral scale, and correlation function.

A. Generation method

The generation method described here is for values of permeability and porosity which are uncorrelated to each other. In generating values of log-permeability or porosity, the autocorrelation function is assumed to be exponential. (Permeability values are obtained by taking the anti-log of log-permeability values.) In addition, the mean, variance, and integral scale value of log-permeability and porosity have to be specified. A covariance matrix can be obtained by combining Eqs. (III-10) and (III-21) through an exponential autocorrelation function. This gives the equation for covariance as

$$C_{xx}(h) = \sigma_x^2 \exp(-h/\lambda_x). \tag{V-1}$$

The covariance matrix, K, is

$$\underline{K} = \begin{bmatrix} C_{xx}(0) & C_{xx}(1) & \cdots\cdots & C_{xx}(n-1) \\ C_{xx}(1) & C_{xx}(0) & \cdots\cdots & C_{xx}(n-2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C_{xx}(n-1) & C_{xx}(n-2) & \cdots\cdots & C_{xx}(0) \end{bmatrix} \qquad \text{(V-2)}$$

where the number in the parenthesis of each element of the covariance matrix is a multiple of lag one (a distance between the center of contiguous blocks). After the covariance matrix is obtained, a set of values of log-permeability or porosity can be generated as follows.[94,95]

It is well known that the covariance matrix K is symmetric and positive-definite. Therefore, it can be decomposed into the product of a lower triangular and an upper triangular matrix where one is the transpose of the other, i.e.

$$\underline{K} = \underline{L}\,\underline{L}', \qquad \text{(V-3)}$$

where $\underline{L}'$ is a transpose of $\underline{L}$ and $\underline{L}$ is a lower triangular matrix. Then, the required random vector, $\underline{y}$ (or set of values of the random variable to be generated), is

$$\underline{y} = \underline{L}\ \underline{w} + \underline{m}, \tag{V-4}$$

where $\underline{w}$ is the vector of an independent normally distributed random number with zero mean and unit standard deviation. The vector $\underline{m}$ is a column vector of specified mean values. In this study, where second-order statistical homogeneity is assumed, all elements of $\underline{m}$ are equal to μ. It can be shown that the mean of $\underline{y}$ is equal to $\underline{m}$ and its variance is equal to $\underline{K}$ as follows:

$$E\{\underline{y}\} = E\{\underline{L}\ \underline{w} + \underline{m}\} = \underline{L}\ E\{\underline{w}\} + \underline{m} = 0 + \underline{m} = \underline{m},$$

and

$$\begin{aligned} \mathrm{Cov}(\underline{y},\underline{y}') &= E\{[\underline{y} - \underline{m}][\underline{y} - \underline{m}]'\} \\ &= E\{(\underline{L}\ \underline{w})(\underline{L}\ \underline{w})'\} \\ &= E\{\underline{L}\ \underline{w}\ \underline{w}'\ \underline{L}'\} \\ &= \underline{L}\ E\{\underline{w}\ \underline{w}'\}\ \underline{L}' \\ &= \underline{L}\ \underline{L}' = \underline{K} \end{aligned}$$

because the mean of the vector $\underline{w}$ is zero and

$$E\{\underline{w}\ \underline{w}'\} = \underline{I},$$

where $\underline{I}$ is the identity matrix.

This method has been called the matrix-equation method by Scheuer and Stoller.[95] Recently, Davis[96] has reintroduced this method for generating values of variates (or random variables) for conditional simulations. This method can be used to generate values of variates for one-, two-, or three-dimensional systems. Davis[97] has shown how this method may be applied for two-dimensional systems. Also, Alabert[98] has used this method to study lead concentration in the Dallas area.

Concerning the decomposition of the symmetric and positive-definite $\underline{K}$ matrix, an alternative method can be used (see Ref. 99). This method does not require the taking of square roots as in the case of the Cholesky method. Davis and Grivet[100] have presented a FORTRAN subroutine for this alternative method.

For a one-dimensional system with the simple form of an autocorrelation function like the one in Eq. (III-12) (for a discrete system), a method based on conditional distribution, which is more efficient than the matrix-equation method for this particular case, can be used. This conditional distribution method has also been presented by Scheuer and Stoller.[95]

The vector $\underline{y}$ was generated using the matrix-equation method described above. The sample mean ($\overline{x}$), sample variance (s^2), sample autocovariance ($c(k)$), and sample intergral scale (λ) were estimated for each realization. The estimated mean values of $\overline{x}$, s^2, $c(k)$, and λ were also evaluated for all realizations. A maximum of 200 realizations was used. It turns out that, except for $\overline{x}$, the values of s^2, $c(k)$, λ, and their estimated means were lower than the input values. This still happened when the seed for the random number generator was changed. Therefore, it was thought that the uniform random generator used had some defect. Consequently, many other random number generators were tried. The random number generators used were:

1. The uniform random number generator from the book by Forsythe *et al.*[101] which was developed in accordance with a suggestion by Knuth.[102] The normal random numbers were generated from the uniform random numbers using the polar method of Box and Muller.[103]
2. The uniform random number generator URAND from CSLIB of the University of Alberta computing system with Box and Muller's algorithm.
3. The uniform random number generator GGUBS with Box and Muller's algorithm.
4. The normal random number generator GGNPM.
5. The multivariate normal random deviate generator (for a given covariance matrix) GGNSM.

The last three generators are in the IMSL library. All of these generators failed to give estimated mean values of s^2, c(k), and $\bar{\lambda}$ close to the input values. To understand why this was the case, a detailed investigation of random numbers and their generators was undertaken.

B. Random number generator

One of the earliest random number generators, proposed and used by von Neuman,[104] is the middle-square random number generator. This generator was later found to be a poor source of random numbers. It generates random numbers that have a short cycle, i.e. a short cycle of repeating elements.[102,105,106]

The most popular random number generators in use today are based on the congruential method. These generators have the following form:

$$X_{n+1} = (aX_n + c) \bmod m, \qquad n > 0, \qquad\qquad (V\text{-}5)$$

If $c = 0$, Eq. (V-5) is called a multiplicative congruential generator. For $c \neq 0$, it is called a mixed congruential generator. All congruential generators have a period less than or equal to m. The period is the maximum number of random numbers which can be produced without repetition. To achieve a maximum period which is equal to m, there are restrictions on the values of a, c, and m. Because a period can be as large as m, it is desirable to choose m as large as possible. However, other considerations have to be taken into account too. See Knuth[102] for recommendations on choosing *a*, *c*, and *m*.

It is important that uniform random numbers be[105]

1. Equidistributed: the numbers should be evenly distributed over the interval [0, 1].
2. Uncorrelated: the correlation between random numbers should be minimum.

These two properties are requirements of randomness. Franklin[107] and, later, Knuth[102] have discussed in detail the randomness of the random number sequence.

There are many methods for testing the randomness of the random sequence. It is recommended that these methods be undertaken before the random number sequence is put into further use. However, these tests are not foolproof and most authors have recommended that the random number sequence be tested with a problem for which the solution is known and which requires essentially the same characteristics as the problem at hand. If the answer obtained from using the random number sequence is close to the true answer, it is likely that the random number sequence can be used satisfactorily with the problem at hand. (The latter test is used, in this study, for testing the validity of the generated values.)

Other methods of generating random numbers have also been presented in the literature. Details of these methods can be found in texts on random numbers and Monte Carlo methods.

After investigating the characteristics of pseudo-random numbers, it was suspected that it might be the correlation in the uniform random number sequence (which is supposed to be independent and random) which caused the sample variance and sample autocovariance estimated from the generated values to be lower than the input values. However, many random number generators have been tried and the results are all in the same direction. It is unlikely that all well-known generators will fail to do the work (although it is possible, see Refs. 101 and 102 about algorithm RANDU).

It was, then, thought that the formulae for estimation of variance and autocovariance might not be appropriate. Both biased and unbiased estimates of autocovariance suggested by Jenkins and Watts[51] have been used. However, the estimate of autocovariance was still much lower than the input value. Consequently, it may be the correlation of values of a variate that causes this phenomenon.

C. Effect of correlation

To investigate the effect of correlation of values of a variate on s^2 and $c(k)$, the literature in the water resources field has been reviewed. In this area of study there are many papers concerning the generation of synthetic flow sequences (see, e.g. Refs. 108-112).

Wallis and O'Connell[113] have reviewed many forms for estimating autocorrelation. All of them are biased estimates of autocorrelation when the samples (or generated values of a random variable) are correlated. The values of the estimate of autocorrelation are smaller than the true values. The smaller the number of samples is, the larger the difference between the estimated and the true values of autocorrelation. (See also Ref. 114.) Wallis and O'Connell[113] have mentioned also the corrections for bias proposed by Marriott and Pope,[115] Kendall,[116] and Quenouille.[117] However, all these bias corrections are only approximate ones.

For a simple autoregressive process,

$$X_i = \rho X_{i-1} + \delta_i, \tag{V-6}$$

where ρ is a lag one correlation and δ_i is a white-noise function consisting of uncorrelated values, Matalas[108] has shown that the expectation of the sample variance is

$$E\{s^2\} = \sigma^2\left[1 - \frac{2}{n(n-1)}\left\{\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\right\}\right], \tag{V-7}$$

where

$$s^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})^2, \tag{V-8}$$

and

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}x_i, \tag{V-9}$$

where n is the number of samples and σ^2 is the variance of X. From Eq. (V-7) it can be seen that if there is no correlation, i.e. $\rho = 0$, s^2 is an unbiased estimate of σ^2. However, when there is a correlation, i.e. $\rho \neq 0$, the term in the brackets on the right hand side of Eq. (V-7) is less than unity. This implies that, when there is serial correlation, $E\{s^2\}$ is less than σ^2 (for a finite value of n). For a given value of ρ, if n approaches infinity, i.e. if there are a large number of samples in the sequence, s^2 tends to be an unbiased estimate of σ^2. On the other hand, when n is small $E\{s^2\} < \sigma^2$, and the smaller n is, the greater the difference between $E\{s^2\}$ and σ^2. This is consistent with the results obtained in this study.

O'Connell[118] has presented an expression for $E\{s^2\}$ similar to that of Eq. (V-7). He has also referred to one of his previous studies[119] where he mentioned the method used to obtain the small sample expectation of the estimate of lag one autocorrelation, $E\{\hat{\rho}\}_n$, where $\hat{\rho}$ is an estimate of ρ. He has also presented a table of values of $E\{\hat{\rho}\}_n$ for various sample sizes. In the table (not shown here), it can be seen that as the sample size is increased, the values of $E\{\hat{\rho}\}_n$ become closer to ρ. This table was obtained through Monte Carlo methods.

It is clear now that there is nothing wrong with the random number generators used. The estimated values (from the generated values of a variate) of $E\{s^2\}$ and $E\{c(k)\}$ are smaller than the input values (σ^2 and $C(k)$) because of the correlation between generated values and because of the sample size. The normal random number generator GGNPM from the IMSL library was selected for further use.

So far there seems to be no exact correction for the bias of the estimate of lag one autocorrelation for a small sample size.[120] There have been arguments about whether an unbiased or a minimum mean square error (minimum value of expectation of the square of differences between the estimated and the true values) is the better criterion for estimation of autocorrelation. However, this is not discussed here because the purpose of this investigation is to set up a criterion for deciding whether the generated values of a variate represent the process used to generate them. It was decided that if the estimated values (estimated from the generated values of variate) of $\mathbf{E}\{s^2\}$ and $\mathbf{E}\{c(k)\}$ are close to the values of $\mathbf{E}\{s^2\}$ and $\mathbf{E}\{c(k)\}$, respectively, the generated values of the variate would be accepted as representative values of the generating process. To use this criterion, the expressions for $\mathbf{E}\{s^2\}$ and $\mathbf{E}\{c(k)\}$ are required. The expression for $\mathbf{E}\{s^2\}$ has been presented by Matalas[108] and O'Connell[118] (see Eq. (V-7)). However, there appear to be no expressions for $\mathbf{E}\{c(k)\}$ (where the autocorrelation has exponential or power form, see Eqs. (III-11) and (III-12)) available in the literature. Therefore, it was decided to derive the expressions for $\mathbf{E}\{c(k)\}$.

It should be noted that this is the only test that was undertaken to check the validity of the generated values of a variate. It was assumed that the normal random number generator GGNPM generated sequences of independent normally distributed random numbers with zero mean and unit standard deviation, as it was supposed to. However, it can be seen that, because (as will be shown later) all the generated values of the variate passed the test undertaken in this study, the generator GGNPM did, in fact, generate acceptable sequences of random numbers.

In this study, the expression for the sample variance, s^2, is the same as that in Eq. (V-8) and the expression for the sample autocovariance of lag k is

$$c(k) = \frac{1}{(n-k)} \sum_{i=1}^{n-k} (x_i - \bar{x})(x_{i+k} - \bar{x}), \qquad k=1,2,\dots,n-1. \tag{V-10}$$

For the assumptions used in this study the expression for $E\{s^2\}$ is the same as the one in Eq. (V-7) (see Appendix D). The derivation of $E\{c(k)\}$ is presented in Appendix D and the final results are:

for $k < n/2$,

$$E\{c(k)\} = \sigma^2\rho^k\Big[1 - \frac{1}{\rho^k}\Big\{\frac{1}{n} + \frac{2\rho}{n(1-\rho)} - \frac{2\rho[k(1-\rho^n)+n\rho^k(1-\rho^{n-2k})]}{n^2(n-k)(1-\rho)^2}\Big\}\Big], \tag{V-11}$$

for k = n/2,

$$E\{c(k)\} = \sigma^2\rho^k\left[1 - \frac{1}{n^2\rho^k}\left\{\frac{n(1-\rho^2)-2\rho(1-\rho^n)}{(1-\rho)^2}\right\}\right], \qquad \text{(V-12)}$$

and for k > n/2,

$$E\{c(k)\} = \sigma^2\rho^k\left[1 - \frac{1}{\rho^k}\left\{\frac{1}{n} + \frac{2\rho}{n(1-\rho)} - \frac{2\rho[k(1-\rho^n)-n\rho^{n-k}(1-\rho^{2k-n})]}{n^2(n-k)(1-\rho)^2}\right\}\right]. \qquad \text{(V-13)}$$

In Eqs. (V-11), (V-12), and (V-13), $\sigma^2\rho^k$ is an autocovariance at lag k (see Eq. (III-12)). The terms in the brackets in Eqs. (V-11) to (V-13) make E{c(k)} different from the input autocovariance. The terms in the brackets will be called correction terms. These terms are dependent on ρ, k, and n.

The correction terms in the brackets in Eq. (V-11) are shown in Tables V-1 to V-4. It turns out that the correction terms for sample autocovariance are the same for the same value of input integral-scale ratio (the ratio of integral scale to total length of the system under consideration, $\lambda' = \lambda/L$) provided that the integral scale is about 5 times larger than the block size. This means that the integral-scale ratio is the one parameter that can be used instead of two parameters, n and ρ, to identify bias in the estimation of autocovariance. This remark is true for processes whose autocorrelation functions have exponential

Table V-1

Correction terms for sample autocovariance: $\lambda/L = 0.05$

x/L	n=40	ρ=.607	n=50	ρ=.670	n=100	ρ=.819	n=200	ρ=.905	n=500	ρ=.961
	lag	corr.	lag	corr.	lag	corr.	lag	corr.	lag	corr.
0*		.926		.922		.914		.909		.907
.005							1	.894		
.010					1	.883	2	.883	5	.883
.015							3	.870		
.020			1	.854	2	.856	4	.856	10	.856
.025	1	.837					5	.840		
.030					3	.823	6	.823	15	.823
.035							7	.804		
.040			2	.780	4	.782	8	.783	20	.783
.045							9	.760		
.050	2	.728			5	.733	10	.734	25	.734
.055							11	.705		
.060			3	.670	6	.673	12	.674	30	.674
.065							13	.639		
.070					7	.600	14	.601	35	.601
.075	3	.550					15	.559		
.080			4	.506	8	.511	16	.512	40	.512
.085							17	.460		
.090					9	.401	18	.403	45	.403
.095							19	.340		
.100	4	.255	5	.261	10	.268	20	.270	50	.271
.105							21	.193		
.110					11	.106	22	.108	55	.109
.115							23	.014		
.120			6	-.104	12	-.093	24	-.090	60	-.089
.125	5	-.229					25	-.205		

* variance

Table V-2

Correction terms for sample autocovariance: $\lambda/L = 0.2$

	n=10	ρ=.607	n=20	ρ=.779	n=40	ρ=.882	n=50	ρ=.905	n=100	ρ=.951	n=200	ρ=.975
x/L	lag	corr.	lag	corr.	lag	corr.	lag	corr.	lag	corr.	lag	corr.
0*		.744		.713		.696		.693		.686		.683
.01									1	.660	2	.661
.02							1	.640	2	.641	4	.641
.03									3	.620	6	.620
.04							2	.597	4	.598	8	.598
.05			1	.572	2	.574			5	.575	10	.575
.06							3	.550	6	.551	12	.551
.07									7	.526	14	.526
.08							4	.499	8	.499	16	.499
.09									9	.471	18	.471
.10	1	.427	2	.438	4	.441	5	.442	10	.442	20	.442
.11									11	.412	22	.412
.12							6	.379	12	.380	24	.380
.13									13	.346	26	.346
.14							7	.310	14	.311	28	.311
.15			3	.269	6	.273			15	.274	30	.274
.16							8	.235	16	.235	32	.235
.17									17	.195	34	.195
.18							9	.152	18	.152	36	.153
.19									19	.108	38	.108
.20	2	.036	4	.055	8	.060	10	.061	20	.061	40	.062
.21									21	.013	42	.013
.22							11	-.039	22	-.038	44	-.038
.23									23	-.092	46	-.092
.24							12	-.149	24	-.148	48	[illegible].147
.25			5	-.214	10	-.209			25	-.206	50	-.206

* variance

Table V-3

Correction terms for sample autocovariance: $\lambda/L = 0.5$

x/L	n=10	ρ=.819	n=20	ρ=.905	n=40	ρ=.951	n=50	ρ=.961	n=100	ρ=.980	n=200	ρ=.990
	lag	corr.	lag	corr.	lag	corr.	lag	corr.	lag	corr.	lag	corr.
0*		.475		.454		.443		.441		.437		.434
.01									1	.418	2	.419
.02							1	.404	2	.404	4	.404
.03									3	.389	6	.389
.04							2	.374	4	.374	8	.374
.05			1	.358	2	.359			5	.359	10	.359
.06							3	.344	6	.344	12	.344
.07									7	.328	14	.329
.08							4	.313	8	.313	16	.313
.09									9	.297	18	.297
.10	1	.275	2	.279	4	.280	5	.280	10	.281	20	.281
.11									11	.264	22	.264
.12							6	.247	12	.248	24	.248
.13									13	.231	26	.231
.14							7	.213	14	.214	28	.214
.15			3	.195	6	.196			15	.196	30	.196
.16							8	.178	16	.179	32	.179
.17									17	.161	34	.161
.18							9	.143	18	.143	36	.143
.19									19	.125	38	.125
.20	2	.099	4	.104	8	.106	10	.106	20	.106	40	.106
.21									21	.087	42	.087
.22							11	.068	22	.068	44	.068
.23									23	.049	46	.049
.24							12	.030	24	.030	48	.030
.25			5	.008	10	.010			25	.010	50	.010
.26							13	-.010	26	-.010	52	-.010

* variance

Table V-4

Correction terms for sample autocovariance: $\lambda/L = 0.8$

	n=10	ρ=.882	n=20	ρ=.939	n=40	ρ=.969	n=50	ρ=.975	n=100	ρ=.988	n=200	ρ=.994
x/L	lag	corr.	lag	corr.	lag	corr.	lag	corr.	lag	corr.	lag	corr.
0*		.344		.329		.321		.320		.316		.315
.01									1	.302	2	.302
.02							1	.291	2	.291	4	.291
.03									3	.280	6	.280
.04							2	.269	4	.269	8	.269
.05			1	.257	2	.258			5	.258	10	.258
.06							3	.247	6	.247	12	.247
.07									7	.235	14	.235
.08							4	.224	8	.224	16	.224
.09									9	.212	18	.212
.10	1	.197	2	.200	4	.201	5	.201	10	.201	20	.201
.11									11	.189	22	.189
.12							6	.178	12	.178	24	.178
.13									13	.166	26	.166
.14							7	.154	14	.154	28	.154
.15			3	.142	6	.142			15	.143	30	.143
.16							8	.131	16	.131	32	.131
.17									17	.119	34	.119
.18							9	.107	18	.107	36	.107
.19									19	.095	38	.095
.20	2	.079	4	.082	8	.083	10	.083	20	.083	40	.083
.21									21	.071	42	.071
.22							11	.059	22	.059	44	.059
.23									23	.047	46	.047
.24							12	.034	24	.034	48	.034
.25			5	.021	10	.022			25	.022	50	.022
.26							13	.010	26	.010	52	.010
.27									27	-.002	54	-.002

* variance

form.

With small numbers of blocks the correction terms for both s^2 and $c(k)$ are slightly different for different numbers of blocks at a specified value of input integral-scale ratio. The correction terms for s^2 and $c(k)$ decrease as the number of blocks increases (Tables V-1 to V-4). (The correction terms for both s^2 and $c(k)$ asymptotically approach constant values as the number of blocks becomes larger.) Because of these differences in values of s^2 and $c(k)$ and the discrete evaluation of sample integral-scale ratio values, the analytical values of the sample integral-scale ratio are slightly different for different numbers of blocks, provided that the numbers of blocks are not large enough (Table V-5). (The words, "analytical values of the sample integral-scale ratio," are used here in order to distinguish them from the values of the sample integral-scale ratio obtained by using Monte Carlo methods which will be called the observed values.) The analytical values of the sample integral-scale ratio, $\hat{\lambda}'_{anal}$, are evaluated using

$$\hat{\lambda}'_{anal} = \frac{\hat{\lambda}_{anal}}{L} = \frac{\Delta s}{L} \sum_{k=0}^{n-1} \frac{E\{c(k)\}}{E\{s^2\}}, \qquad \text{(V-14)}$$

where L is the length of the system, Δs is the length of each block, $c(k)$ is the sample autocovariance of lag k, and s^2 is the sample variance. Only positive values of $c(k)$ are used in Eq. (V-14).

Table V-5

Analytical values of sample integral-scale ratio

n	10	20	40	50	100	200	500
$\Delta x/L$	.1	.05	.025	.02	.01	.005	.002
$(\lambda/L)_i$	$(\lambda/L)_s$	$(\lambda/L)_s$	$(\lambda/L)_s$	$(\lambda/L)_s$	$(\lambda/L)_s$	$(\lambda/L)_s$	$(\lambda/L)_s$
0.800	0.168	0.143	0.132	0.130	0.126	0.124	0.123
0.700	0.167	0.141	0.130	0.128	0.124	0.122	0.121
0.600	0.164	0.139	0.128	0.126	0.122	0.120	0.119
0.500	0.161	0.135	0.124	0.122	0.118	0.116	0.115
0.400	0.157	0.130	0.119	0.117	0.113	0.111	0.110
0.300	0.150	0.123	0.111	0.109	0.105	0.103	0.102
0.200	0.137	0.110	0.098	0.096	0.091	0.089	0.088
0.100	0.118	0.086	0.073	0.070	0.065	0.063	0.062
0.090	0.115	0.082	0.069	0.066	0.061	0.059	0.058
0.080	0.112	0.078	0.065	0.062	0.057	0.055	0.053
0.070	0.108	0.075	0.060	0.058	0.053	0.050	0.049
0.060	0.105	0.070	0.055	0.053	0.048	0.045	0.044
0.050	0.101	0.065	0.050	0.047	0.042	0.040	0.038
0.040	0.100	0.060	0.044	0.042	0.036	0.034	0.032
0.030	0.100	0.056	0.038	0.035	0.030	0.027	0.026
0.020	0.100	0.051	0.032	0.029	0.023	0.020	0.018
0.010	0.100	0.050	0.026	0.022	0.015	0.012	0.010
0.008	0.100	0.050	0.025	0.021	0.013	0.010	0.008
0.005	0.100	0.050	0.025	0.020	0.011	0.008	0.006
0.001	0.100	0.050	0.025	0.020	0.010	0.005	0.002

$(\lambda/L)_i$ is input λ/L
$(\lambda/L)_s$ is sample λ/L

It can be seen from Tables V-1 to V-4 that the correction terms for sample autocovariance become negative after a certain lag size. From the results obtained for the highest value of input integral-scale ratio ($\lambda' = 1.0$) used, the maximum length of lag before the correction term becomes negative is about one-fourth of the total length of the system. The smaller the value of the input integral-scale ratio is, the shorter the distance (compared to the total length) at which the correction term becomes negative. This means that $E\{c(k)\}$ will become negative after some distance from the origin. If the sample integral-scale ratio is estimated from the positive values of $E\{c(k)\}$ versus length, the sample integral-scale ratio will always be less than the input integral-scale ratio (for the generated system of a finite number of blocks). This is shown in Figure V-1.

From all the numerical results obtained, the values of the correction terms in Eqs. (V-12) and (V-13) are always negative. This may imply that it is not possible to obtain a correct estimate of autocovariance for a lag larger than one-half of the length of the series if only one series is used. This actually conforms with the common recommendation in time series that the estimation of autocovariance should be restricted to lags that are less than one-fourth of the length of the series.[121]

Now let us look at the values of sample integral-scale ratio compared to the input values. Table V-5 reveals many interesting features as follows:

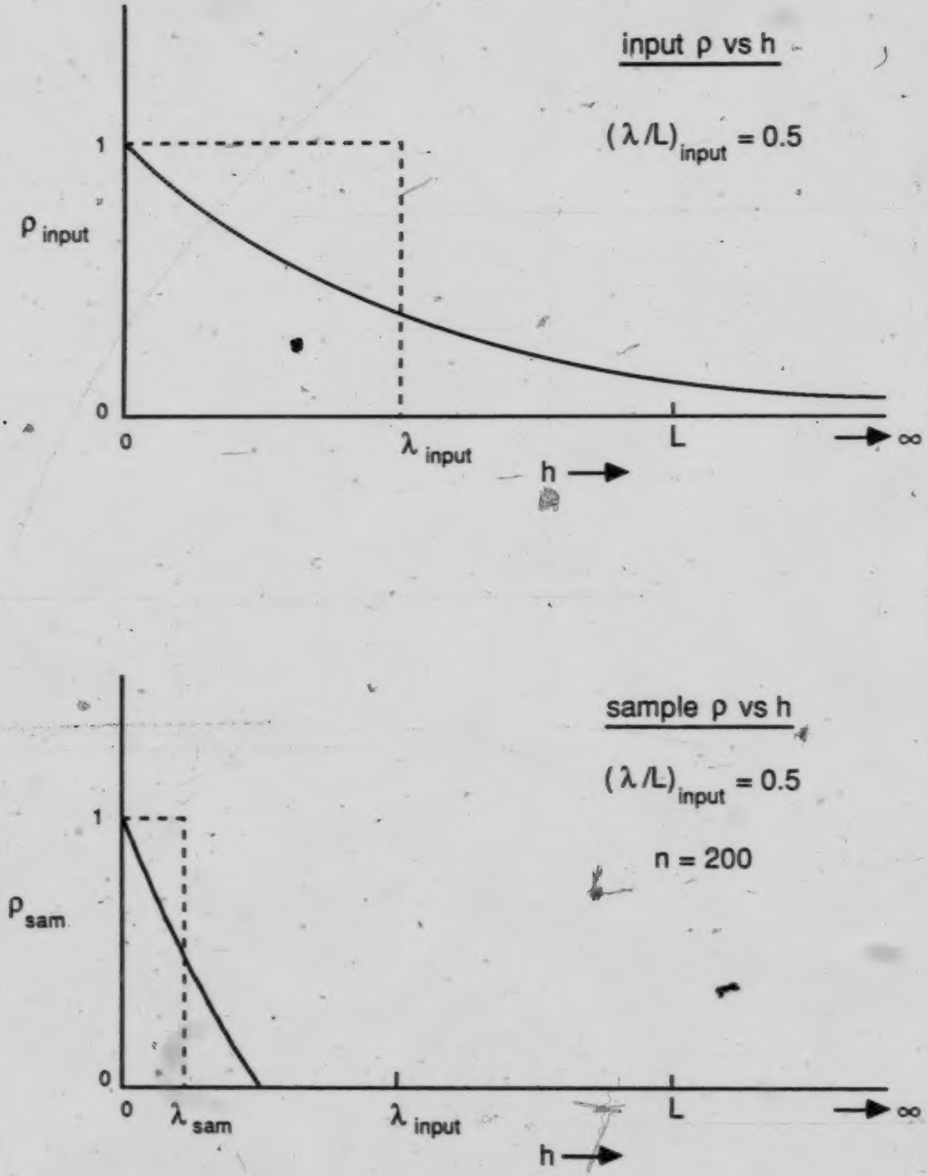


Figure V-1 Difference between input and sample integral scale.

1. The analytical values of the sample integral-scale ratio, λ'_{anal}, are always less than the input integral-scale ratio values as long as the input integral scale values are greater than two times the size of the block. (Note: the analytical values of the sample integral-scale ratio are the values of the sample integral-scale ratio estimated from $E\{s^2\}$ and $E\{c(k)\}$ (see Eq. (V-14)).)
2. When the input integral scale values are less than two times the size of the block, the values of λ'_{anal} tend to be greater than the input values and as the input values of integral scale decrease, the analytical values of the sample integral scale approach the values of the block size. This is theoretically correct because the value of an integral scale smaller than the block size is undefined.
3. For the values of input integral-scale ratio equal to or greater than 0.5, the values of λ'_{anal} are only slightly different for different values of the input integral-scale ratio and/or for a different number of blocks. These differences might be difficult to distinguish when values of the variate are generated by using random number generators. (Note that the results discussed here are from the use of Eqs. (V-7) and (V-12) to (V-13).)
4. All upper values of λ'_{anal} in Table V-5 are lower than 0.2. The maximum values of λ'_{anal} become lower as the

number of blocks increases. The value of λ'_{anal} asymptotically approaches a constant value as the number of blocks becomes large.

Based on these remarks it is recommended that the values of the input integral scale used be between the size of the block and one-half of the total length of the system. Input values of integral scale outside this range should not be used.

The upper limit on values of λ'_{anal}, mentioned in Remark (4) above, may be the reason that the integral-scale ratio values used by Smith and Freeze[20,21] and by Smith and Brown[49] are almost always lower than 0.2. The maximum values of the integral-scale ratio used by Smith and Freeze[20,21] are: for one-dimensional systems, $\lambda'_{max} = 0.09$ with typical values of 0.02 to 0.04 (numbers of blocks used are 100 and 200); and for two-dimensional systems, $\lambda'_{max} = 0.13$ in the x-direction, and $\lambda'_{max} = 0.27$ in the z-direction, with typical values of 0.08-0.1 in the x-direction and 0.15-0.18 in the z-direction. (The number of blocks used is 20 in the x-direction and 10 in the z-direction.) Smith and Brown[49] used the following values: for one-dimensional systems, $\lambda'_{max} = 0.15$ with typical values of 0.08-0.12 (the number of blocks used is 20); and for two-dimensional systems, $\lambda'_{max} = 0.07$ in the x-direction (20 blocks). (It is not clear what the length was in the z-direction; therefore, λ' in the z-direction could not be evaluated.) Generally, these authors used low values of the integral-scale ratio in the range shown in Table V-5. The unusually high value of λ'_{max} used by Smith and Freeze[21] may be caused by the discrete estimation of the integral scale and the small number of blocks. These authors obtained values of the integral scale by discrete estimation of integration of the sample autocorrelation versus distance, and they used a first-order nearest-neighbor model in generating values of the variate. By using this model for one- and two-dimensional systems, the functional form of the autocorrelation function will not be exponential. Therefore, the remarks above are speculative only.

Because the values of λ'_{anal} in Table V-5 are not much different for different numbers of blocks (with equal values of input integral-scale ratio) and for different values of input integral-scale ratio (with equal number of blocks), Tables V-6 and V-7 were constructed so as to check if the observed values of the sample integral-scale ratio, λ'_{obs}, can be distinguished for the case of a different number of blocks and for the case of different values of input integral-scale ratio. (λ'_{obs} is evaluated using a formula similar to the one for λ'_{anal} except that the estimated values (from the generated values of variate) of $E\{c(k)\}$ and $E\{s^2\}$ are used instead of $E\{c(k)\}$ and $E\{s^2\}$ themselves.) Note here that only cases with a small number of blocks (approximately less than 500 blocks for magnitudes of the integral-scale ratio considered here) are considered. This is useful because use of a large number of blocks for a Monte Carlo

Table V-6

Analytical and observed values of sample integral-scale ratio for different numbers of blocks
input $\lambda/L = 0.2$ (200 realizations)

	n = 10	n = 20	n = 40	n = 50
Analytical	0.137	0.110	0.098	0.096
Observed	0.138	0.106	0.098	0.098

Table V-7

Analytical and observed values of sample integral-scale ratio for different values of input integral-scale ratio
n = 50 (200 realizations)

input (λ/L)	anal. (λ/L)	obs. (λ/L)
0.800	0.130	0.132
0.700	0.128	0.130
0.600	0.126	0.128
0.500	0.122	0.124
0.400	0.117	0.119
0.200	0.096	0.098
0.100	0.070	0.071
0.050	0.047	0.048

study is usually too expensive. Consequently, use of a small number of blocks is generally necessary. This is especially true in this study.

Table V-6 shows that for the input integral-scale ratio value of 0.2 with 200 realizations, the values of λ'_{obs} for a different number of blocks can be distinguished. This is true for a system having a small number of blocks. For a system with a large number of blocks, both values of λ'_{anal} and values of λ'_{obs} are difficult to distinguish (for a different number of blocks.) The values of λ'_{obs} for this case are remarkably close to λ'_{anal} values.

Table V-7 shows various values of λ'_{obs} for an equal number of blocks (50 blocks, 200 realizations). Again the values of λ'_{obs} are remarkably close to λ'_{anal} values. From the results of this table it is hoped that the values of λ'_{obs} (estimated from the generated values of the variate) can be distinguished for different values of the input integral-scale ratio. However, as previously stated, for an input integral-scale ratio greater than 0.5, both λ'_{anal} and λ'_{obs} values (for different values of input integral-scale ratio) are only slightly different, and differentiation of the values would be very difficult.

It should be noted that λ'_{anal} in Eq. (V-14) is a biased estimate of input integral-scale ratio. This bias is due to two causes: the bias of s^2 and $c(k)$; and the discrete nature of the system (finite number of blocks). Consequently, the correction term for λ'_{anal} is likely to be complicated in nature. Therefore, it is not recommended that Table V-5 be used for converting the values of the sample integral-scale ratio to the values of the input integral-scale ratio. The main purpose of presenting Table V-5 is to demonstrate that the values of the sample integral-scale ratio can be much different than the values of the input integral-scale ratio.

So far it has been shown that the values of sample variance and sample autocovariance (s^2 and $c(k)$) from the generated values of the variate are smaller than the input values. This is because s^2 and $c(k)$, Eqs. (V-8) and (V-10), are biased. As the value of the input integral-scale ratio increases, the differences between the sample and input values of variance and autocovariance become larger. Similarly, the values of sample integral-scale ratio, evaluated from the generated values of a random variable, are smaller than the input values. Now when the generated values of log-permeability and porosity are used in the flow equations, and the resulting pressure and saturation are calculated, which values of the integral scale and variance of log-permeability and porosity should be used to describe the results, the input or the sample values? From the discussion above it is clear that the input values should be used. This is because the generated values of log-permeability and porosity actually represent the process with the given input values of variance and integral scale. The sample values are different from the input values because of the bias of the estimates, Eqs. (V-8) and (V-10).

The controlling parameter of the bias seems to be the magnitude of the input integral-scale ratio itself.

Therefore, in this study the input values of variance and integral-scale ratio are reported with the resulting pressure and saturation. And to assure that the generated values of the variate really represent the input process with a specified variance and integral-scale ratio, the estimated values of $E\{s^2\}$ and $E\{c(k)\}$ (evaluated from all realizations) are compared to the values of $E\{s^2\}$ and $E\{c(k)\}$ (Eqs. (V-7), (V-11) to (V-13)). The generated values of the variate are accepted if the difference between the observed bias (estimated value of $E\{\overline{x}\}$, $E\{s^2\}$, or $E\{c(k)\}$ minus μ, σ^2, or $C(k)$, respectively) and the analytical bias ($E\{\overline{x}\}$, $E\{s^2\}$, or $E\{c(k)\}$ minus μ, σ^2, or $C(k)$, respectively) is less than twice the standard error (of estimated mean of $\overline{x}$, s^2, or $c(k)$, respectively) (see Ref. 115). If this is not achieved, the number of realizations is increased until this criterion is met or a new set of random numbers is used. For all generated values of log-permeability used, this criterion was achieved. Table V-8 shows an example of the test results for the case with $\sigma_Y = 0.8$ and $\lambda_Y' = 0.3$ (only positive sample autocovariances are shown). The number of realizations used is 400. It can be seen that all the differences between the observed biases and the analytical biases are less than twice the standard errors.

Let us investigate the effect of variance and integral-scale ratio on the resulting pressure and

Table V-8

Test results for the case with $\sigma_Y = 0.8$, $\lambda'_Y = 0.3$

	est. mean	obs. bias	anal. bias	S.E. of est. mean	est. variance
$\overline{Y}$	-29.9891	-0.0424	0	0.0270	0.2905
s^2	0.3869	-0.2531	-0.2548	0.0131	0.0686
c(1)	0.2627	-0.2791	-0.2817	0.0119	0.0571
c(2)	0.1768	-0.2818	-0.2874	0.0108	0.0463
c(3)	0.1060	-0.2822	-0.2913	0.0088	0.0313
c(4)	0.0437	-0.2849	-0.2934	0.0070	0.0195

saturation. If the variance is high, the range of values of the variate will be large. This will make the variation of values of the variate between realizations high. It is expected that the variation of values of the resulting pressure and saturation at each point would be high too. For the case of integral-scale ratio, if the integral-scale ratio is high, the variation of values of the variate within each realization should be low. However, there should be a high variation in values of the variate between realizations. This would cause high variation in values of the resulting pressure and saturation. This is plausible because the higher the spatial correlation of the random variable, say permeability, the less information is known about the probabilistic properties of permeability from one realization. With less information about permeability, less is known about the resulting pressure and saturation. In other words, the less information about permeability is known, the more uncertainty there is concerning information on the resulting pressure and saturation. This is consistent with the results obtained by Smith and Freeze.[20]

A recent study by Russo and Jury[122,123] on two-dimensional systems has shown that bias in estimation of the integral scale is dependent on the number of sample points and the magnitude of the integral scale itself.

D. Generation of correlated permeability and porosity

In this section some form of correlation between permeability and porosity is assumed. To take into account the correlation between permeability and porosity in the generation of values of the variates of these two parameters, a slightly different scheme than previously described is needed. Before presenting the method of generation of values of permeability and porosity which are correlated to each other, let us consider the appropriate form of correlation for these two parameters.

It has been concluded that for a highly uncorrelated pore network, there is a strong linear relationship between log-permeability and total porosity, whereas for correlated networks, a relationship between permeability and total porosity is poor.[82] The uncorrelated pore network occurs when throats are randomly associated with pores and the correlated pore network occurs when large pores are associated with large throats and small pores with small throats.

A major cause of uncorrelated pore networks is diagenetic clay.[81] The studies by Yuan[82] and Wendt *et al*.[80] confirm this conclusion. Figures V-2, V-3, and V-4 show the linear trend of log-permeability versus porosity for an uncorrelated-pore-network rock. (In Figure V-4 Zone 3 represents a correlated-pore network, i.e., a network poorly cemented, high gravel; and, consequently, there is a poor relationship between permeability and porosity.)

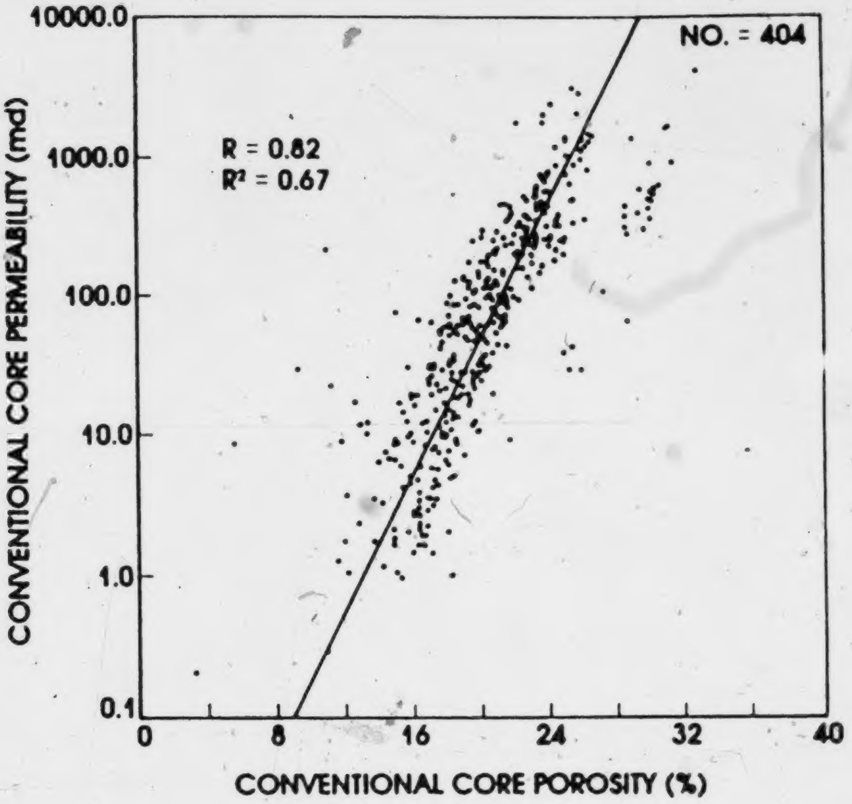


Figure V-2 Porosity versus permeability of conventional core plugs from Miocene sandstones, Pattani basin, Gulf of Thailand (from Ref. 148).

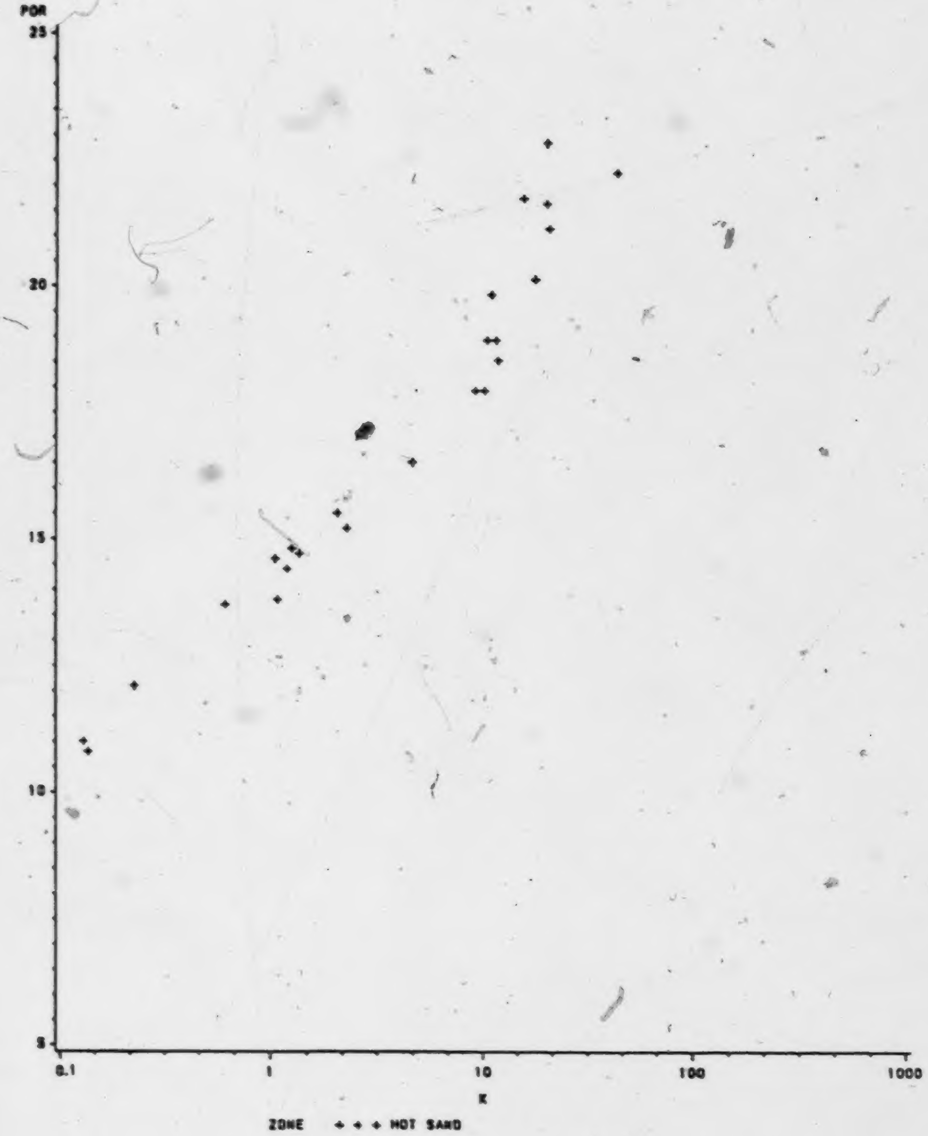


Figure V-3 Core porosity versus permeability of North Sea sandstones (from Ref. 82).

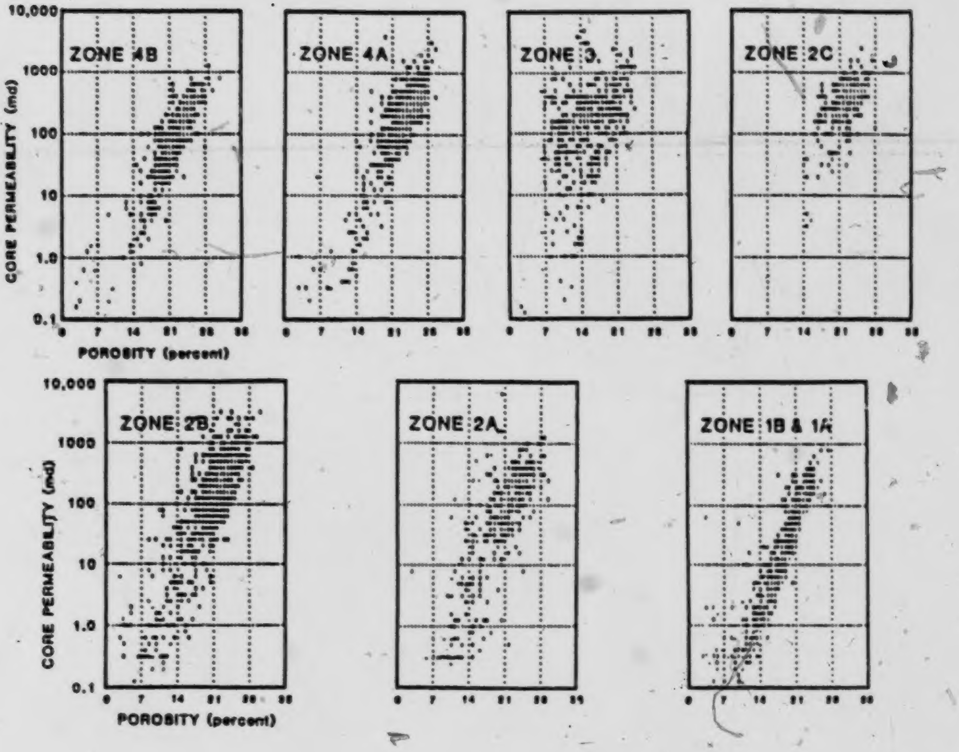


Figure V-4 Porosity-permeability plot by zone, Prudhoe Bay (from Ref. 80).

In this study it is assumed that the porous medium used has an uncorrelated-pore-network property. Therefore, it is justified to assume a linear relationship between log-permeability and total porosity. To generate values of permeability (or log-permeability) and porosity, first the log-permeability values are generated for given values of standard deviation and integral scale of log-permeability using the generation method described earlier. (The computer code for generation of permeability values is given in Appendix E.) Porosity values for each location are evaluated from

$$\ln(k) = a + b\phi, \qquad \text{(V-15)}$$

where k is permeability, a and b are constants which can be obtained from field data, and ϕ is porosity. Implicit in the use of Eq. (V-15) is the assumption that there is an exact relationship between permeability and porosity. This is equivalent to use of the Kozeny equation which gives an exact relation between permeability and porosity. However, it should be noted that porosity in Eq. (V-15) is treated as a random variable. This is because permeability is a random variable and a and b are constants. Because of the exact relationship between permeability and porosity, the variation in porosity is exactly related to the variation in log-permeability.

VI. NUMERICAL MODEL

Numerical methods are used to solve the two-phase flow equations. The numerical model was developed for immiscible, incompressible two-phase flow through a one-dimensional serial system. The governing flow equations for the oil and water phases are

$$\frac{1}{\mu_o}\frac{\partial}{\partial x}\left[Akk_{ro}\left(\frac{\partial p_o}{\partial x} - \rho_o g\frac{\partial D}{\partial x}\right)\right] + \frac{q^*_o}{\Delta x} = A\phi\frac{\partial S_o}{\partial t}, \qquad \text{(VI-1)}$$

and

$$\frac{1}{\mu_w}\frac{\partial}{\partial x}\left[Akk_{rw}\left(\frac{\partial p_w}{\partial x} - \rho_w g\frac{\partial D}{\partial x}\right)\right] + \frac{q^*_w}{\Delta x} = A\phi\frac{\partial S_w}{\partial t}, \qquad \text{(VI-2)}$$

where the subscripts o and w are for oil and water, respectively.

The partial differential equations, Eqs. (VI-1) and (VI-2), were discretized and solved by the implicit pressure - explicit saturation (IMPES) method. Difference equations for evaluation of pressure and saturation were developed and the Thomas algorithm was used in solving the matrix of simultaneous equations. Most techniques for programming were adapted from the work by Fanchi *et al.*[124] The computer code, used in this study, for solving the flow equations is given in Appendix E.

A. Grid system

The point-distributed grid system (see Ref. 125) was used in the formulation of the difference equations. This grid system allowed exact specification of the pressure at the boundary. Figure VI-1 is an example of the grid system used. It was assumed that the one-dimensional system considered consisted of N-2 equal-sized rectangular blocks. These blocks were arranged in the horizontal direction. Values of permeability and porosity for each block were estimated using the variate generation method described in the previous chapter. The values were controlled by the given values of mean, standard deviation, and integral scale of log-permeability. At both ends of the system, each of the boundary blocks was divided into two unequal-size blocks. Therefore, in the formulation of the difference equations, the first and the second (simulation) blocks had the same rock properties. Similarly, the last block and the second-last block also had the same rock properties. (The total number of simulation blocks was equal to N.)

B. Initial and boundary conditions

Initial conditions for all cases considered were as follows:

1. The initial water saturation was fixed at the irreducible water saturation.
2. The initial pressure was 0.1 MPa.

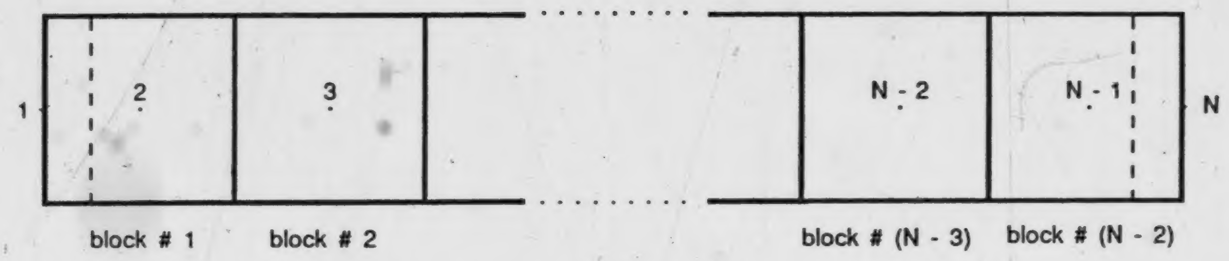


Figure VI-1 The point-distributed grid system used in this study.

In this study two types of boundary conditions were considered:

1. constant total flow rate and constant pressure at the outlet end,
2. constant pressure at both ends.

It was assumed that flow occurred at both ends while the other boundaries were closed. At the boundaries it was assumed that there was no end effect. The oil and water flow rates at the outlet end were evaluated by keeping the water saturation of the Nth block equal to that of the (N-1)th block.

C. Flow conditions

Both steady-state and transient flows were considered. Steady-state flow was investigated using both types of boundary conditions, while transient flow was investigated for the system with constant total flow rate and constant pressure at the outlet end only. In the case of steady-state flow, the ratio of the oil and water flowing into the inlet end was held constant. The flow continued until the proportions of oil to water at the inlet and outlet ends were the same. The criterion for reaching the steady-state condition was that the ratio of the difference between the inlet and outlet water flow rates to the inlet water flow rate was equal to 0.00001. When this criterion was met the simulator run was stopped. For transient flow, the run was stopped at a specific time. All total flow rates used for both steady-state and transient conditions were such that the instability number was smaller than π^2 (see Ref. 126). This constraint was imposed to ensure that the flow was stable.

In the development of the simulator, many empirical relationships were used. These relationships were for relative permeabilities to oil and water, capillary pressure, and irreducible water saturation.

D. Relationship for relative permeabilities

Many investigators[71,127-134] have proposed expressions for relative permeabilities related to porous medium properties, capillary pressure, and/or saturation. Brutsaert[135] gave a good review on this subject. The relationships for relative permeabilities that were used in this study are

$$k_{rw} = \left[\frac{S_w - S_{wirr}}{1 - S_{wirr}}\right]^2 \int_{S_{wirr}}^{S_w} \frac{dS_w}{P_c^2} \bigg/ \int_{S_{wirr}}^{1} \frac{dS_w}{P_c^2}, \qquad \text{(VI-3)}$$

and

$$k_{ro} = \left[1 - \frac{S_w - S_{wirr}}{1 - S_{wirr}}\right]^2 \int_{S_w}^{1} \frac{dS_w}{P_c^2} \bigg/ \int_{S_{wirr}}^{1} \frac{dS_w}{P_c^2}, \qquad \text{(VI-4)}$$

where S_{wirr} is irreducible water saturation and P_c is capillary pressure. Here it is assumed that the porous media considered are isotropic. Burdine[132] analysed experimental data for permeability and capillary suction versus saturation by means of the equation developed from the

Kozeny theory. He found that the tortuosity ratio is approximately a linear function of normalized saturation. From this finding he proposed Eqs. (VI-3) and (VI-4). On the other hand, based on "cutting and rejoining" considerations, Wyllie and Gardner[133] derived equations for relative permeabilities which are identical to Eqs. (VI-3) and (VI-4).

Corey[136] and Brook and Corey[137] suggested that for typical porous media (see Ref. 138), the normalized saturation was inversely proportional to capillary pressure squared, i.e.

$$\frac{1}{P_c^2} = \text{constant} \cdot S_w^*, \qquad \text{(VI-5)}$$

where S_w^* is the normalized saturation,

$$S_w^* = \frac{S_w - S_{wirr}}{1 - S_{wirr}}. \qquad \text{(VI-6)}$$

Eq. (VI-5) was suggested for drainage processes and is good for $P_c > P_d$ and $S_w > S_{wirr}$, where P_d is the displacement pressure. Wyllie and Gardner[133] plotted many sets of data which approximately obey the relationship in Eq. (VI-5).

Corey[136] used the expression in Eq. (VI-5) in Eqs. (VI-3) and (VI-4) and obtained

$$k_{rw} = (S_w^*)^4, \qquad \text{(VI-7)}$$

and

$$k_{ro} = (1-S_w^*)^2(1-S_w^{*2}). \qquad (VI-8)$$

Eqs. (VI-7) and (VI-8) are derived for drainage processes. Naar and Henderson[139] proposed that the relative permeabilities for imbibition processes be given by the relationship between normalized imbibition saturation and normalized drainage saturation as

$$S_{w,imb}^* = S_{w,dr}^* - \frac{1}{2}S_{w,dr}^{*2}, \qquad (VI-9)$$

(see also Refs. 140 and 141). For imbibition processes, the relative permeability to water can be obtained by using $S_{w,imb}^*$ directly in Eq. (VI-7). To obtain the relative permeability to oil, convert $S_{w,imb}^*$ to $S_{w,dr}^*$ using Eq. (VI-9) and use this $S_{w,dr}^*$ in Eq. (VI-8).

The coefficient, 1/2, used in Eq. (VI-9) was based on an oil-water system with a recovery factor, RF, equal to 0.5. A general form for drainage and imbibition saturations can be obtained by substituting (1-RF) for the 1/2 in Eq. (VI-9).[142] Therefore, Eq. (VI-9) can be written as

$$S_{w,imb}^* = S_{w,dr}^* - (1-RF)S_{w,dr}^{*2}, \qquad (VI-10)$$

Because employing RF=0.5 with the data used in this study gave a narrow range between S_{wirr} and $1-S_{or}$, RF=0.75 was used instead. Although this value of RF is quite high and possibly unrealistic, it is believed that it did not affect the final conclusions in this study. Figure VI-2 shows typical relative permeabilities versus S_w used in the numerical study.

In addition, the capillary pressure curve used in this study was modified from the relationship in Eq. (VI-5) by displacement of P_c down by an amount equal to the displacement pressure, P_d. This allowed P_c at $1-S_{or}$ to be zero which is consistent with imbibition processes. A typical capillary pressure curve used in this study is illustrated in Figure VI-3.

E. Relationship for irreducible water saturation

In order to take into account the effect of heterogeneity on the relative permeabilities an empirical expression which relates irreducible water saturation to permeability and porosity was used. This empirical expression is[143]

$$S_{wirr} = 100 \frac{\phi^{2.25}}{(k)^{1/2}}. \qquad \text{(VI-11)}$$

This equation is applicable for sands, shaly sands, or other hydrocarbon-saturated intergranular rocks.[143] It is based on empirical observations and is similar in form to a general expression proposed by Wyllie and Rose.[144]

By using Eqs. (VI-5), (VI-7), (VI-8), (VI-10), and (VI-11), the effect of heterogeneity of porous media on

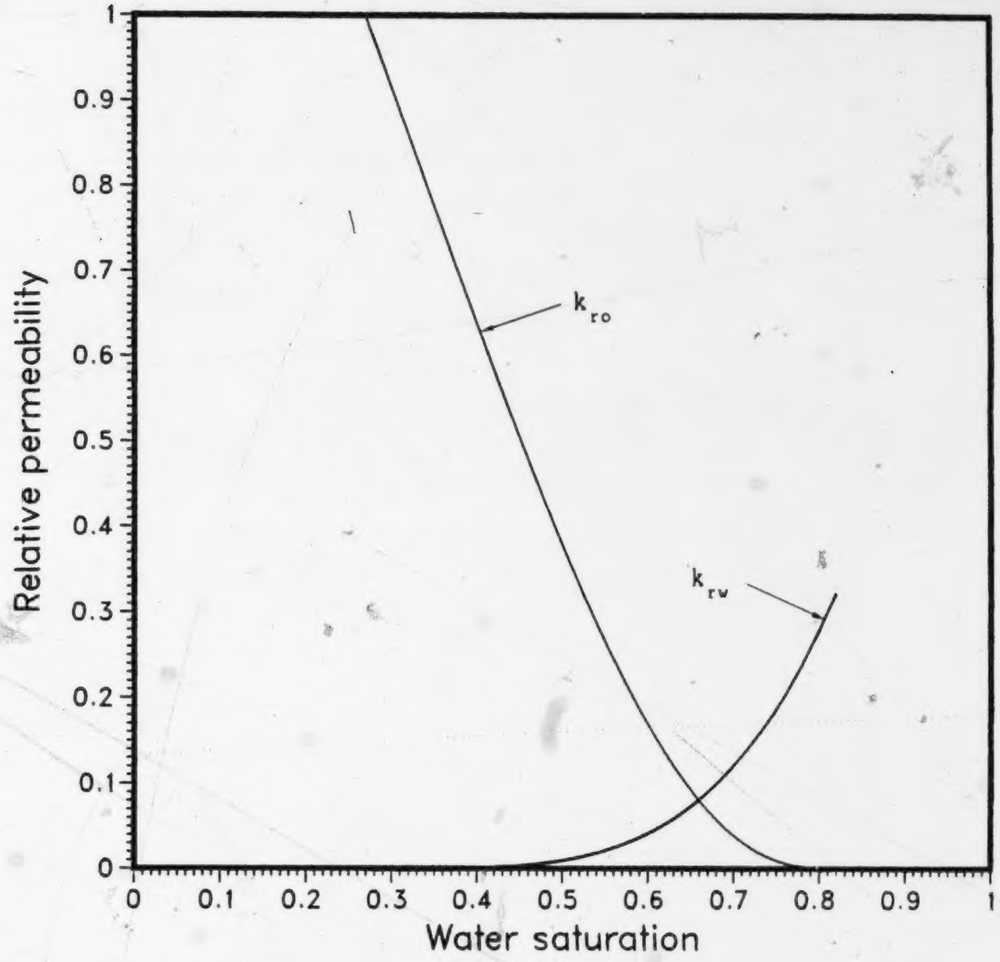


Figure VI-2 Typical relationship between relative permeability and water saturation used in this study.

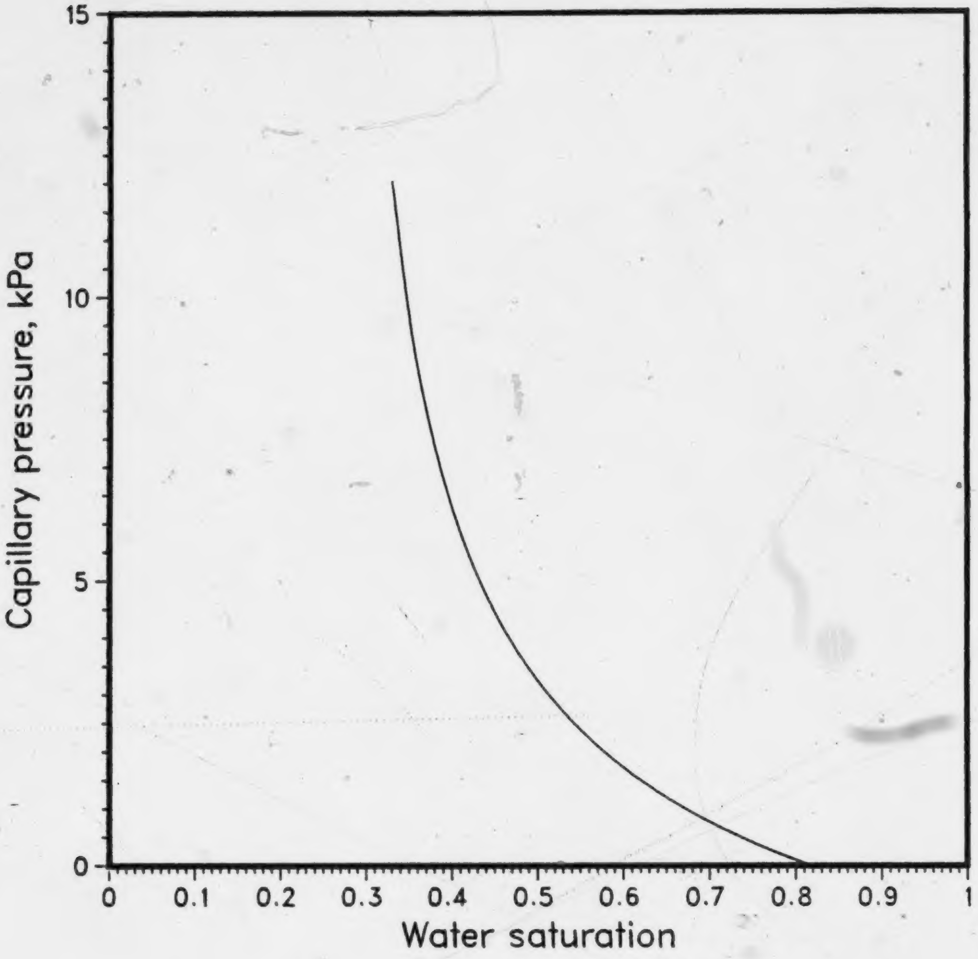


Figure VI-3 Typical capillary pressure curve for imbibition processes used in this study.

relative permeability and capillary pressure was accounted for. In addition, through these empirical formulae, the residual oil saturation was also related to porous medium heterogeneity.

All these empirical relationships are based partly on the theoretical concepts mentioned and partly on field data. It is hoped that these empirical relationships lead to a flow model that closely simulates the flow condition in a real reservoir. It should be emphasized that all the results obtained in this study are based on these empirical relationships. Therefore, the results can only be applied to porous media that can be represented approximately by these empirical relationships.

F. Verification of the simulator

The simulator has been tested for many test conditions. First, it was run at various fractional flow, f_w, values under the steady-state condition. All resulting pressures and saturations obtained were identical to analytical results. Then, using a linear relationship between relative permeabilities and saturation, the calculated saturation profiles were compared to the analytical saturation profiles. Except for some numerical dispersion, which is acceptable, the general comparison is quite good (Figure VI-4). Figure VI-5 shows a saturation profile which was obtained by using Eqs. (VI-7) and (VI-8) for relative permeabilities.

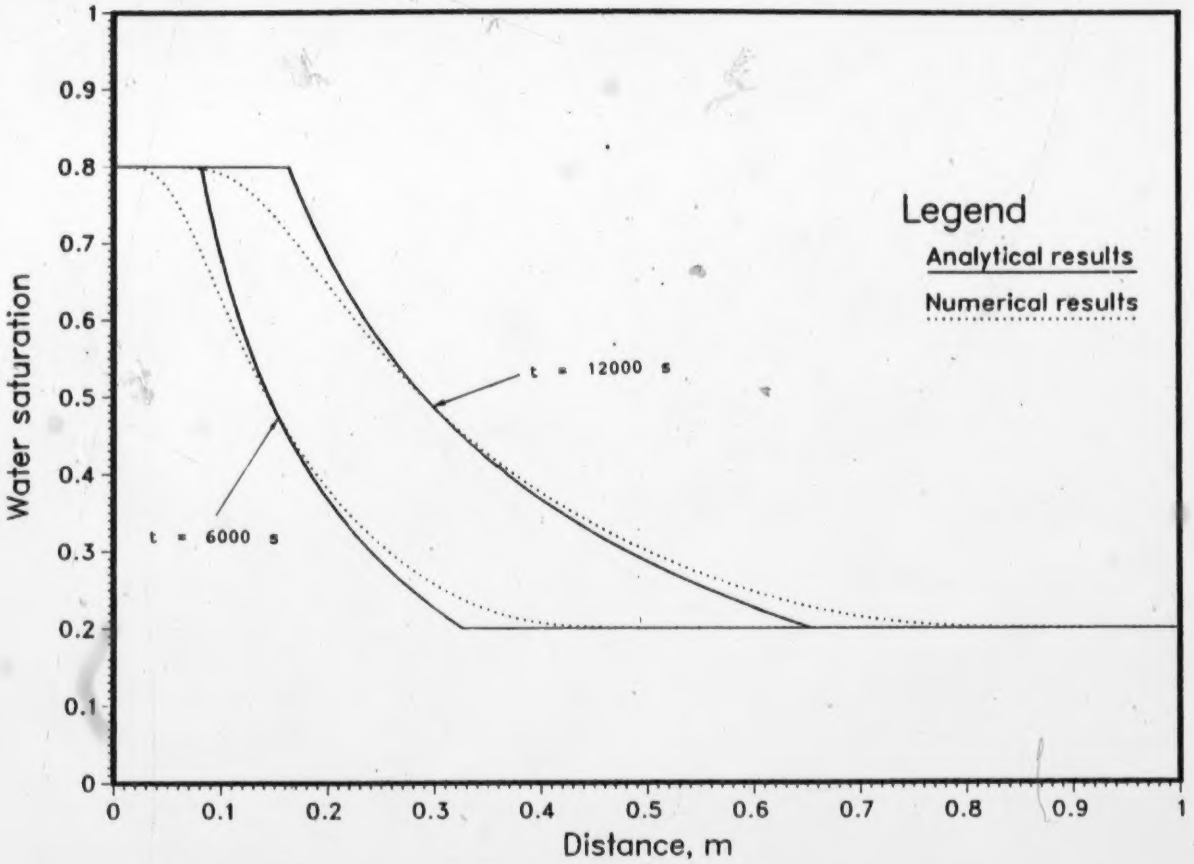


Figure VI-4 Comparison of numerical and analytical results using linear relationship between relative permeability and saturation.

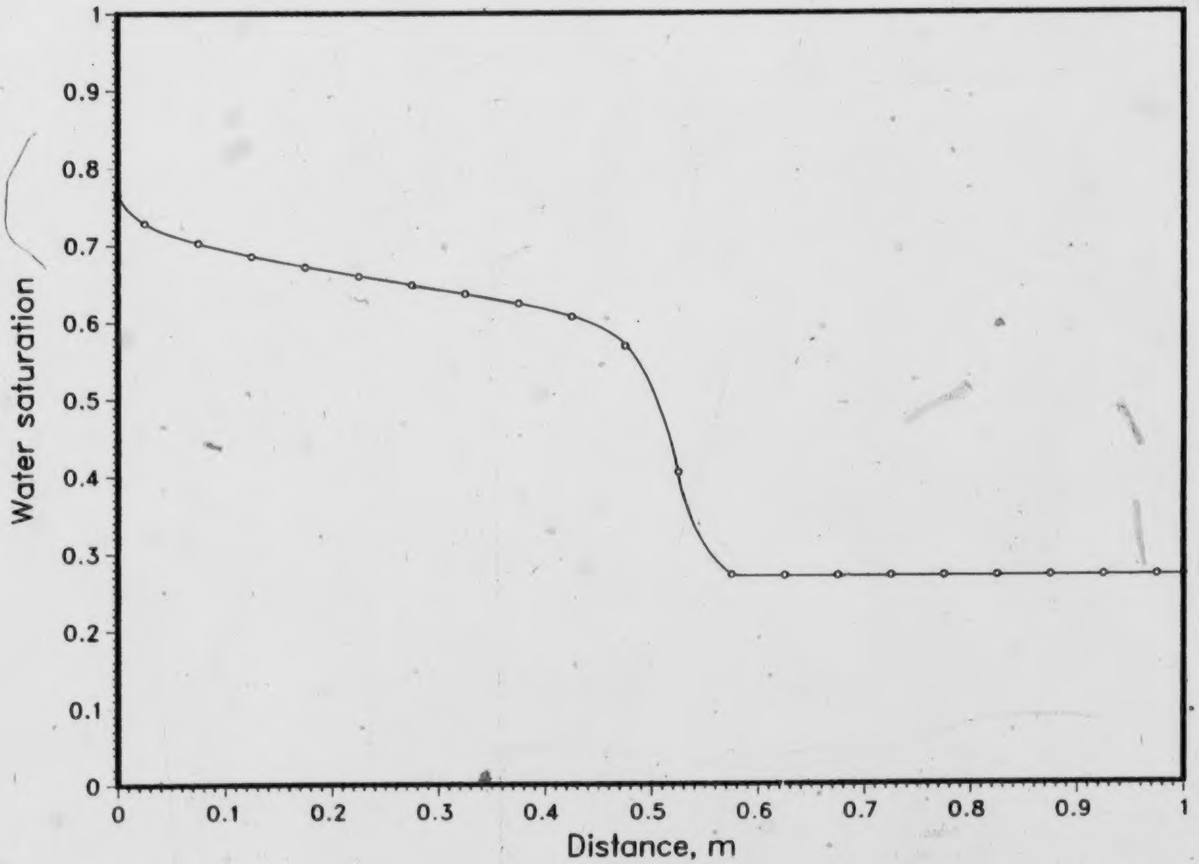


Figure VI-5 Saturation profile from numerical method using Eqs. (VI-7) and (VI-8) for relationship between relative permeability and saturation.

Concerning numerical dispersion problems,[145-147] the simulator was run for systems of twenty and forty blocks. The results for the system consisting of twenty blocks are comparable to the ones for the forty-block system. The smearing of the saturation front for the twenty-block system is tolerable (Figure VI-6). Therefore, for practical considerations, a system of twenty blocks was used in the Monte Carlo simulation runs.

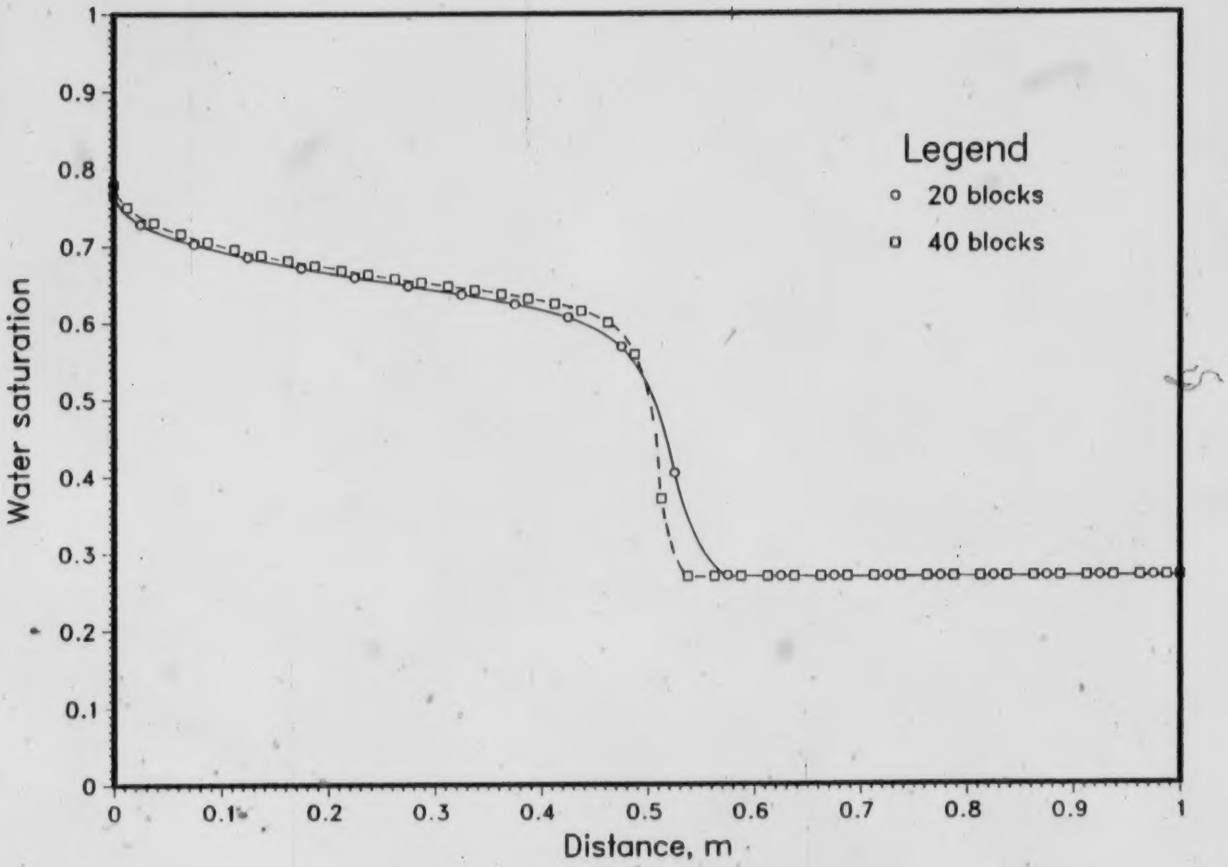


Figure VI-6 Comparison of saturation profile for different number of blocks.

VII. STEADY-STATE FLOW

The main purpose of studying the effect of heterogeneity of rock properties on the resulting pressure and saturation is to investigate whether the concept of effective permeabitity is applicable to two-phase flow systems and to study the effect of standard deviation and integral scale of log-permeability on the mean and standard deviation of pressure and saturation. In addition, the effect of capillary pressure and of mobility ratio is investigated. After presenting the data used in the Monte Carlo simulations, the results for both types of boundary conditions are presented and discussed.

A. Data

To investigate the effect of standard deviation, σ_Y, and integral scale ratio, λ_Y', of log-permeability on the resulting pressure and saturation, sets of values of standard deviation and integral scale ratio are used (Table VII-1). The values of σ_Y range from 0 to 0.8. This range is selected from the data presented by Freeze[4] for sandstone. The λ_Y' values range from 0.05 to 0.3. The 0.05 value corresponds to the block size used (20 blocks in the system). This range of λ_Y' is believed to cover all the practical cases. These sets of values of σ_Y and λ_Y' form the basic data for all the cases studied here.

Table VII-1

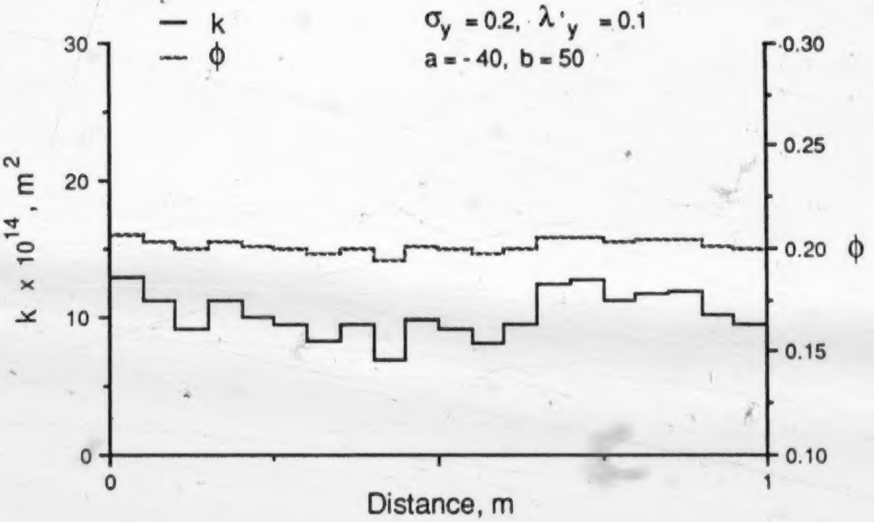
Values of standard deviation and integral scale ratio of log-permeability used in the Monte Carlo runs

Case	σ_Y	λ'_Y
1	homogeneous	
2	0.2	0.05
3	0.5	0.05
4	0.8	0.05
5	0.2	0.10
6	0.5	0.10
7	0.8	0.10
8	0.2	0.30
9	0.5	0.30
10	0.8	0.30

The input value of the mean of log-permeability, μ_Y, is -29.9468. This value corresponds to a permeability value of 9.87×10^{-14} m^2 (100 millidarcies). The constants, a and b, in Eq. (V-15) are estimated from a plot of log-permeability and porosity data for the Gulf of Thailand.[148] These values are: a = -40.0, b = 50.0. Using these values the mean of porosity is equal to 0.201.

The program was developed to generate values of permeability and porosity which are correlated to each other by the relationship in Eq. (V-15). The number of realizations generated for each set of σ_Y and λ'_Y values was 400. The generated values of log-permeability for all sets of σ_Y and λ'_Y values yielded differences, between the observed biases and the analytical biases, which were less than twice the standard errors. Therefore, the generated values were acceptable according to the criterion set up in Chapter V. Examples of generated values of permeability and porosity are shown in Figures VII-1 and VII-2.

For the cases with constant flow rate and constant pressure at the outlet end, a total flow rate of 4×10^{-7} m^3/s was used. This flow rate resulted in an instability number value lower than π^2. For the cases with constant pressure at both ends, the pressure at the inlet end was 10 MPa for the unfavorable case (viscosity ratio = 5.0, end-point mobility ratio = 1.6) and 3 MPa for the favorable case (viscosity ratio = 1.0, end-point mobility ratio = 0.3). In all the cases studied, the outlet pressure was 0.1 MPa.



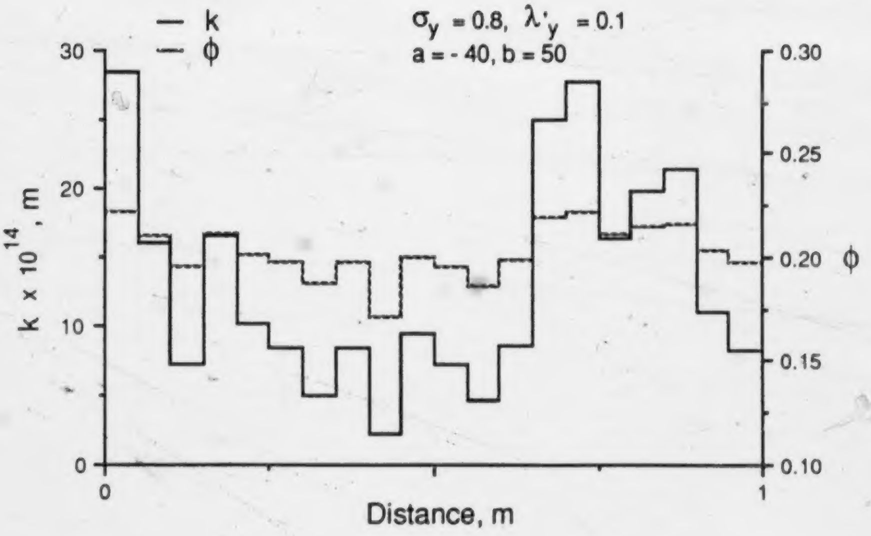


Figure VII-1 Examples of realization for different values of standard deviation of log-permeability.

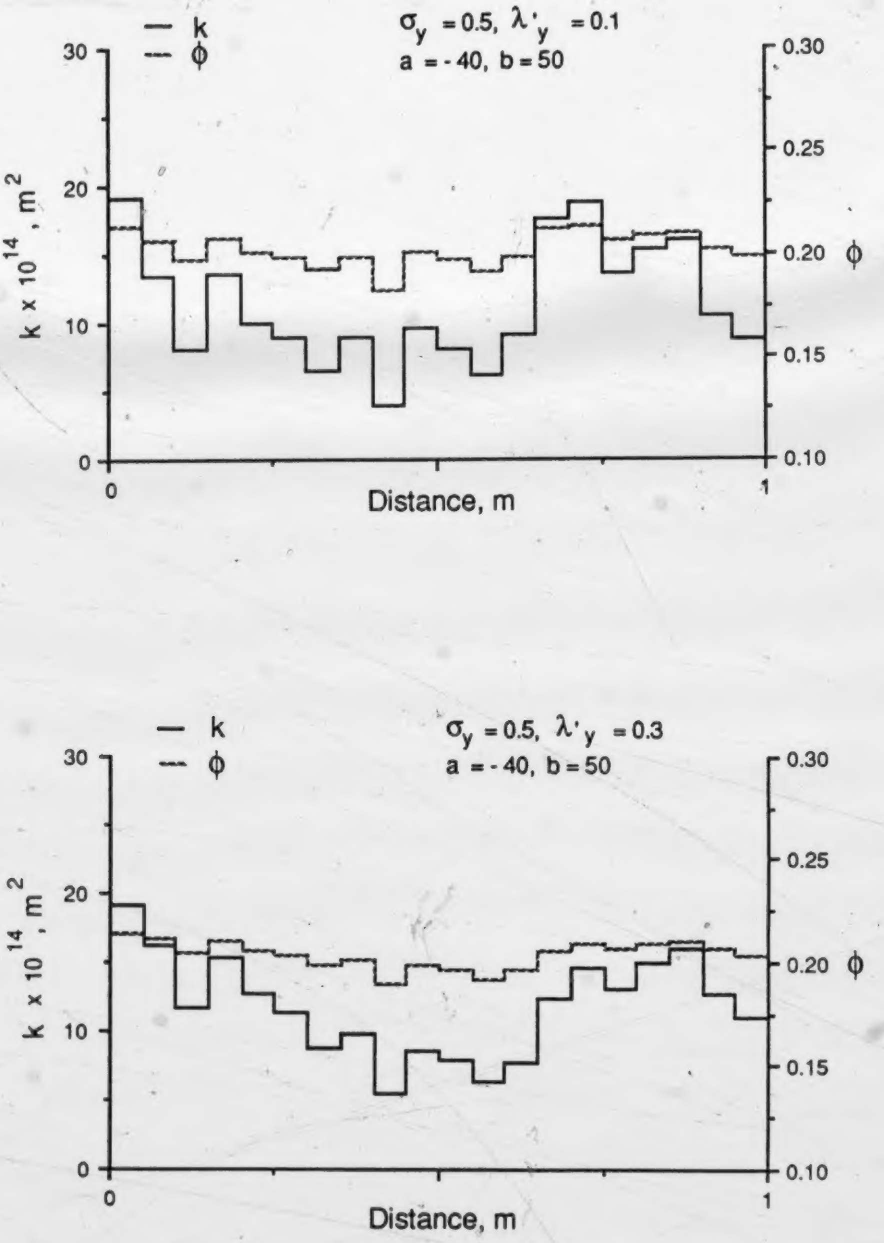


Figure VII-2 Examples of realization for different values of integral scale of log-permeability.

B. Constant flow rate and constant pressure at the outlet end

This type of boundary condition is identical to the one used for the steady-state coreflood tests. The results from this study will give some understanding of the effect of heterogeneity on the relationship between relative permeabilty and saturation, in addition to its effect on the resulting pressure and saturation. In the following the effect of standard deviation and integral scale of log-permeability is presented.

Effect of σ_Y and λ'_Y on the mean oil pressure

Figure VII-3 shows the effect of σ_Y on the mean of the oil pressure, $\bar{p}_o$. The solid diamond-shaped symbol is for the homogeneous case with the permeability equal to k_G or $\exp(\mu_Y)$. It can be seen that, as σ_Y increases, $\bar{p}_o$ increases at all points, except at the outlet end where the pressure is constant and known with certainty. In other words, as σ_Y increases, the pressure gradient increases. In Figure VII-3, the capillary pressure is assumed to be zero; therefore, the effect of σ_Y on the mean of the water pressure, $\bar{p}_w$, will be identical with that of σ_Y on $\bar{p}_o$.

The effect of λ'_Y on $\bar{p}_o$ is shown in Figure VII-4. It seems that λ'_Y has some effect on $\bar{p}_o$ but that the effect is very small. To obtain a clearer picture of the effect of various values of σ_Y and λ'_Y on $\bar{p}_o$, Figure VII-5 is plotted for σ_Y ranging from 0 to 0.8 and λ'_Y ranging from 0.05 to

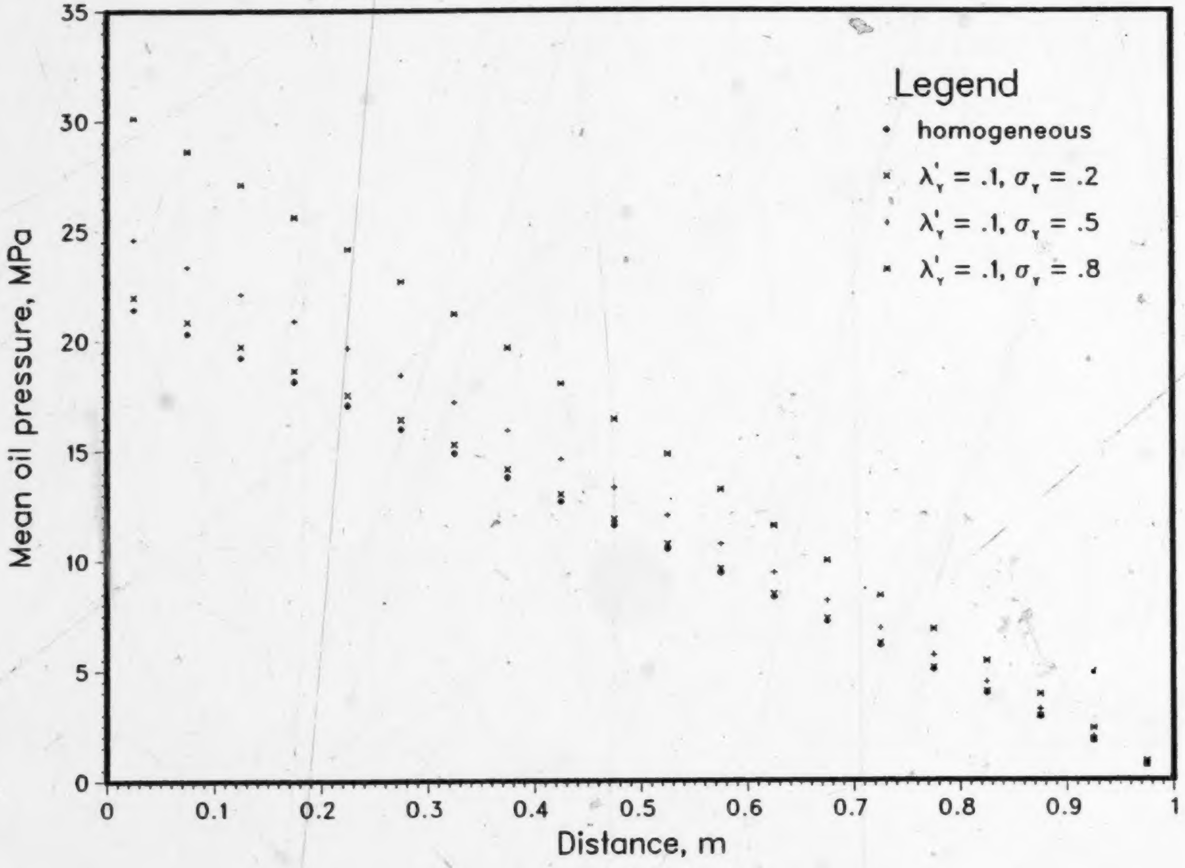


Figure VII-3 Effect of σ_Y on mean oil pressure, $\lambda_Y' = 0.1$

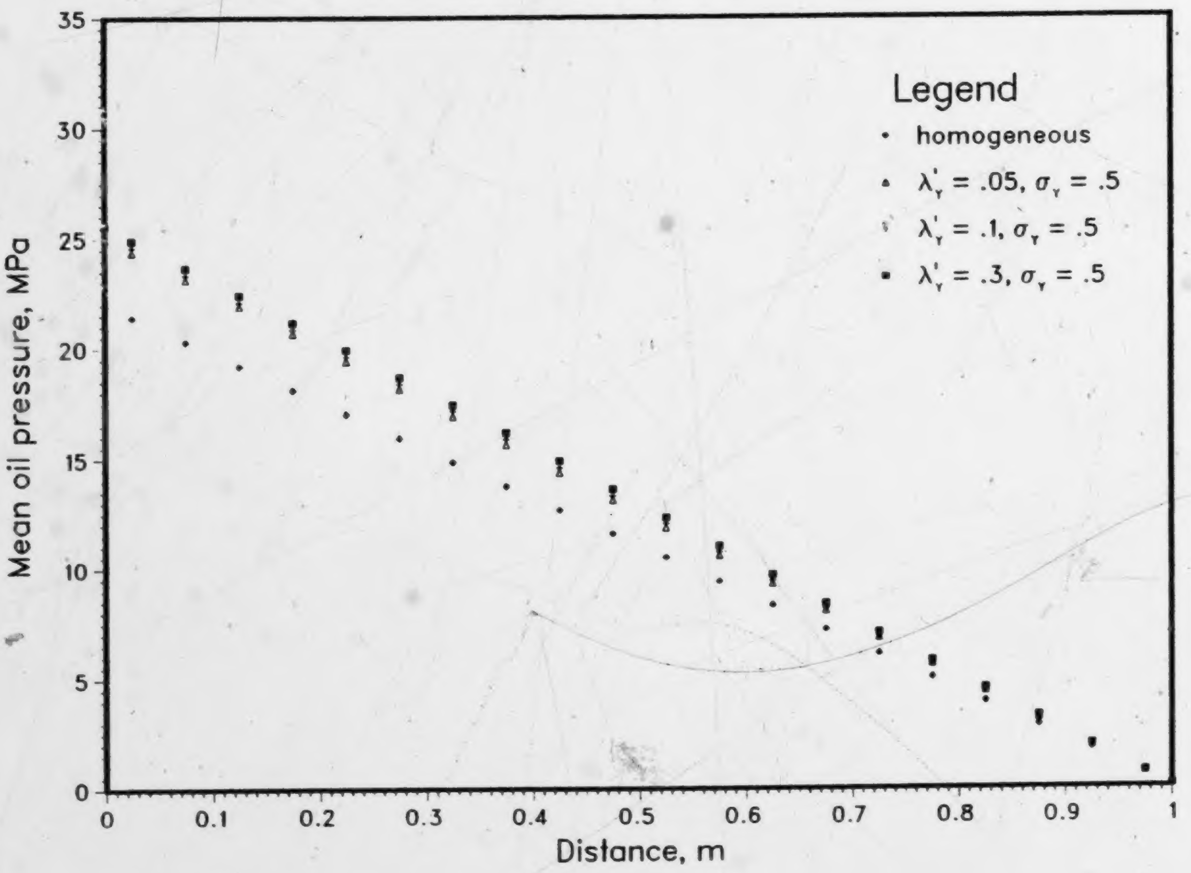


Figure VII-4 Effect of λ'_Y on mean oil pressure, $\sigma_Y = 0.5$

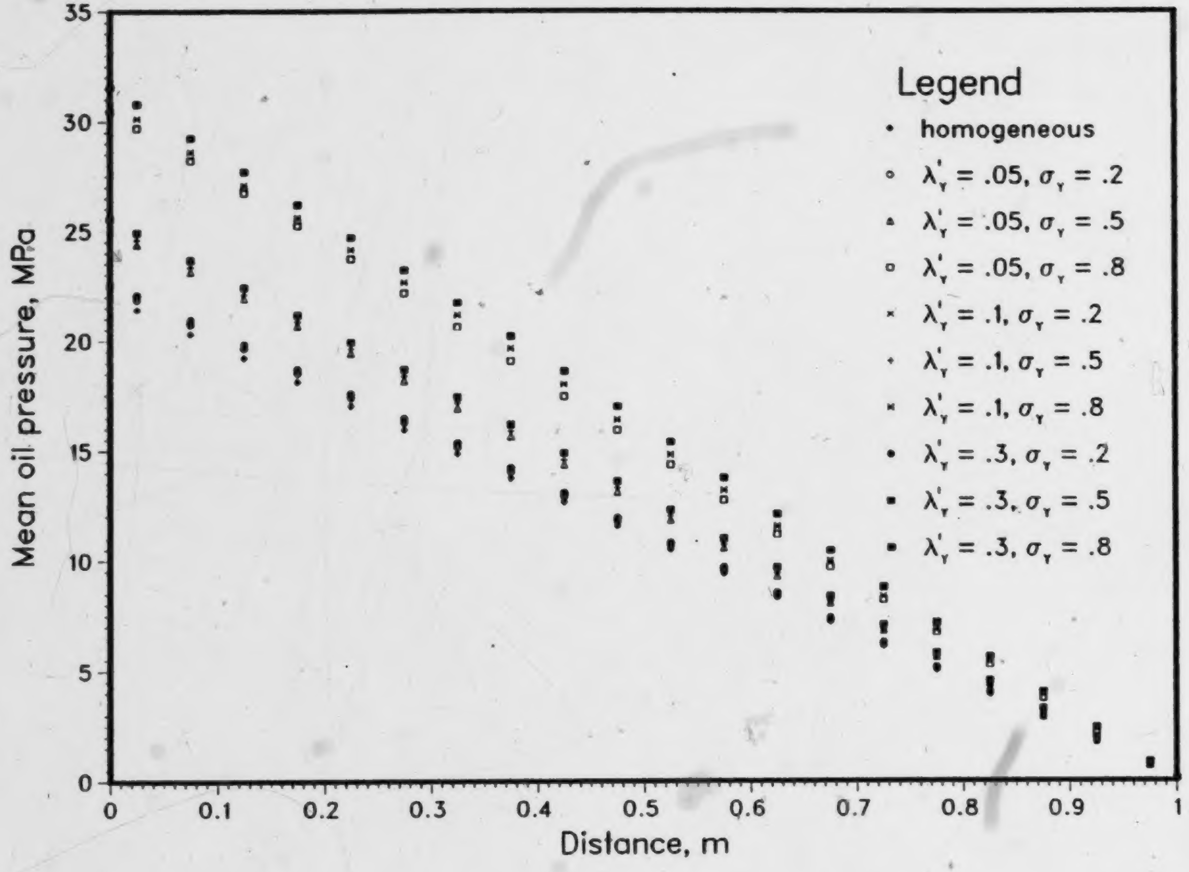


Figure VII-5 Effect of σ_Y and λ'_Y on mean oil pressure.

0.3. It is obvious that σ_Y has a strong effect on $\overline{p}_o$ whereas λ_Y' has a weak effect on $\overline{p}_o$. The effect of λ_Y' on $\overline{p}_o$ appears to be more pronounced at higher values of σ_Y. However, this is not the case as will be shown in the following.

Before considering the effect of λ_Y' on $\overline{p}_o$ for two-phase systems, it would be more appropriate to consider the single-phase systems. In a single-phase system, the saturation is not involved; therefore, the effect of λ_Y' can be investigated alone. Figure VII-6 shows a plot of mean pressure versus distance for various values of σ_Y and λ_Y'. The mean pressure was obtained by using the following relation to estimate the effective permeability,

$$k_{eff,est} = \left[\frac{1}{m} \sum_{i=1}^{m} \frac{1}{k_i^*} \right]^{-1}, \qquad \text{(VII-1)}$$

where

$$k_i^* = \frac{L}{\Delta x} \left[\sum_{j=1}^{n} \frac{1}{k_{ij}} \right]^{-1}, \qquad \text{(VII-2)}$$

and m is the number of Monte Carlo runs, n is the number of blocks in the system, k_{ij} is the permeability for block j and realization i, and k_i^* is the permeability of realization i. Then, the mean pressure gradient is calculated from

$$\frac{\overline{\Delta p}}{\Delta L} = \frac{q\mu}{A k_{eff,est}}. \qquad \text{(VII-3)}$$

From Figure VII-6, it seems that λ_Y' has an effect on the mean pressure, $\overline{p}$. However, in the derivation in Appendix A,

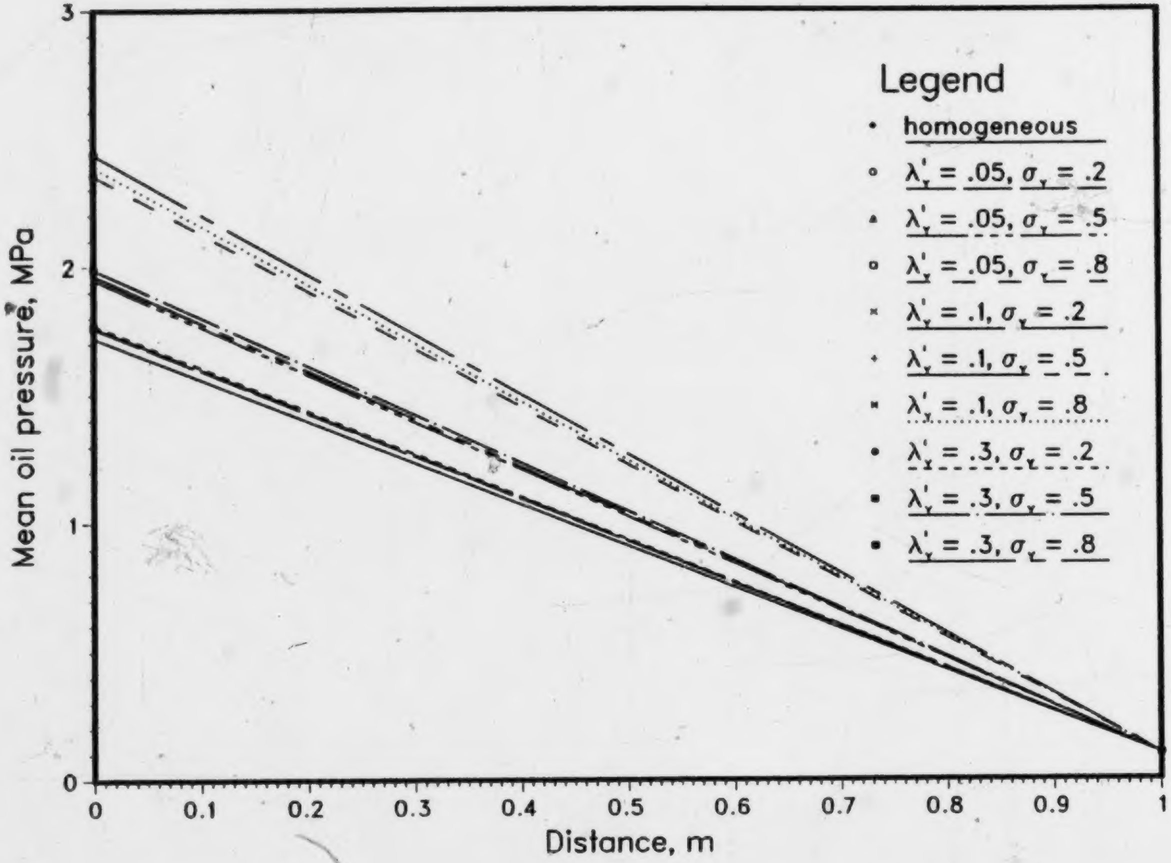


Figure VII-6 Effect of σ_Y and λ'_Y on mean pressure gradient for single-phase flow (based on the inlet and outlet end pressures).

it has been shown that the effective permeability depends only on σ_Y and not on λ_Y'. This leads one to the conclusion that the numerical experimental results are not consistent with the theoretical results. This is rather discouraging because it suggests that either the experimental or the theoretical results are wrong. Many possible sources of error were checked. Scrutiny of the overall mean value of log-permeability for each case revealed that these overall mean values are different from case to case. (The overall mean value is evaluated from the relationship,

$$\overline{Y}_{\text{overall}} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} Y_{ij}, \qquad \text{(VII-4)}$$

where m and n are defined as before and $Y = \ln(k)$.) The values of the overall mean of Y and their percent difference from the input mean value are shown in Table VII-2. This table reveals that even for the case of the highest degree of heterogeneity ($\sigma_Y = 0.8$, $\lambda_Y' = 0.3$) the estimated mean of the generated values of log-permeability is only about 0.14% different from the input mean value. This is quite acceptable from the variate generation point of view. However, for this highest degree of heterogeneity case, the estimated mean of the generated values of log-permeability is -29.9891 which is equivalent to a permeability value of about 9.46×10^{-14} m^2. This permeability value differs from the input permeability (9.87×10^{-14} m^2) by about 4.14%. This difference is not small. When inspecting the values of the

Table VII-2

Comparison of input and overall means of log-permeability

Case	$\overline{Y}_{overall}$	**%diff.***
1	-29.9468	0
2	-29.9490	0.0074
3	-29.9524	0.0188
4	-29.9558	0.0302
5	-29.9526	0.0195
6	-29.9614	0.0489
7	-29.9725	0.0859
8	-29.9574	0.0355
9	-29.9732	0.0883
10	-29.9891	0.1414

$$* \ \%\text{difference} = \frac{\overline{Y}_{overall} - \overline{Y}_{input}}{\overline{Y}_{input}} \times 100\ \%$$

geometric means, $\exp(\overline{Y}_{\text{overall}})$, of cases 4, 7, and 10, which all have $\sigma_Y = 0.8$ and λ'_Y ranges from 0.05 to 0.3, it can be seen that they are not negligible. Therefore, it is quite possible that the difference in mean pressure gradients for cases 4, 7, and 10, which have the same value of σ_Y but different λ'_Y, is due to the difference in geometric mean of the generated values of permeability and not due to the difference in λ'_Y. This is shown in Table VII-3 where the effective permeabilities obtained from the numerical experiment, Eq. (VII-1), are compared with the theoretical value, Eq. (II-3), using both the geometric mean from the input and the overall mean value of log-permeability. The differences in Column VI are much smaller than those in Column IV.

Because the same sets of realizations are used for both single- and two-phase flow, it is highly probable that the differences in mean oil-pressure gradient for different values of λ'_Y at high values of σ_Y are due to the difference in the overall geometric means of the generated variate from the input values, and not due to the effect of λ'_Y.

From scrutiny of Figure VII-5, it can be seen that at high values of σ_Y and λ'_Y, the data points deviate from the straight line connecting the inlet and outlet pressure. This may be due to two causes: the effect of a change in saturation due to high values of σ_Y and λ'_Y and/or an insufficient number of realizations used in the numerical experiment. At high values of σ_Y and λ'_Y, the number of

Table VII-3

Comparison of the experimental and the theoretical values of effective permeability

I Case	II $k^{*}_{eff,exp}$	III $k^{**}_{eff,theo,inp}$	IV $\frac{III-II}{II}\times 100$	V $k^{***}_{eff,theo,over}$	VI $\frac{V-II}{II}\times 100$
1	9.86922×10^{-14}	9.86922×10^{-14}	0	9.86922×10^{-14}	0
2	9.65107×10^{-14}	9.67380×10^{-14}	0.2355	9.65225×10^{-14}	0.0123
3	8.65719×10^{-14}	8.70956×10^{-14}	0.6049	8.66066×10^{-14}	0.0401
4	7.09823×10^{-14}	7.16653×10^{-14}	0.9622	7.10211×10^{-14}	0.0546
5	9.61606×10^{-14}	9.67380×10^{-14}	0.6005	9.61757×10^{-14}	0.0157
6	8.57942×10^{-14}	8.70956×10^{-14}	1.5169	8.58307×10^{-14}	0.0425
7	6.99790×10^{-14}	7.16653×10^{-14}	2.4097	6.98449×10^{-14}	-0.1917
8	9.56761×10^{-14}	9.67380×10^{-14}	1.1099	9.57151×10^{-14}	0.0408
9	8.46609×10^{-14}	8.70956×10^{-14}	2.8758	8.48238×10^{-14}	0.1924
10	6.85145×10^{-14}	7.16653×10^{-14}	4.5987	6.86950×10^{-14}	0.2634

* effective permeability from numerical experiments.

** effective permeability from theoretical relationship, using input geometric mean.

*** effective permeability from theoretical relationship, using overall geometric mean.

realizations needed may be greater than 400 (which is used for all the cases run) in order to cover the whole range of possible values of log-permeability and porosity. In order to eliminate the effect of a change in saturation, the single-phase cases were run again. The plot of mean pressure versus distance for the single-phase cases is shown in Figure VII-7. From an inspection of the data points at high values of σ_Y and λ_Y', it can be seen that these data points deviate from the straight line joining the inlet and outlet pressure in the same direction as the two-phase cases. This leads one to the conclusion that this deviation is due to an insufficient number of realizations and that a change in saturation does not have a significant effect. To confirm this conclusion, the single-phase case with $\sigma_Y = 0.8$ and $\lambda_Y' = 0.3$ was run for 1000 and 2000 realizations. The data points fall closer and closer to the straight line (Figure VII-8) as the number of realizations increases. Table VII-4 shows the percent deviation of the data points from the straight line at points along the system for the various numbers of realizations used.

Another interesting observation is that all mean pressures at the inlet end from the numerical experiments are equal to their counterparts calculated by using the effective permeability concept [Eq. (VII-1)]. This helps to confirm that if the number of realizations is large enough all the data points should fall on the straight line joining the inlet and outlet pressures.

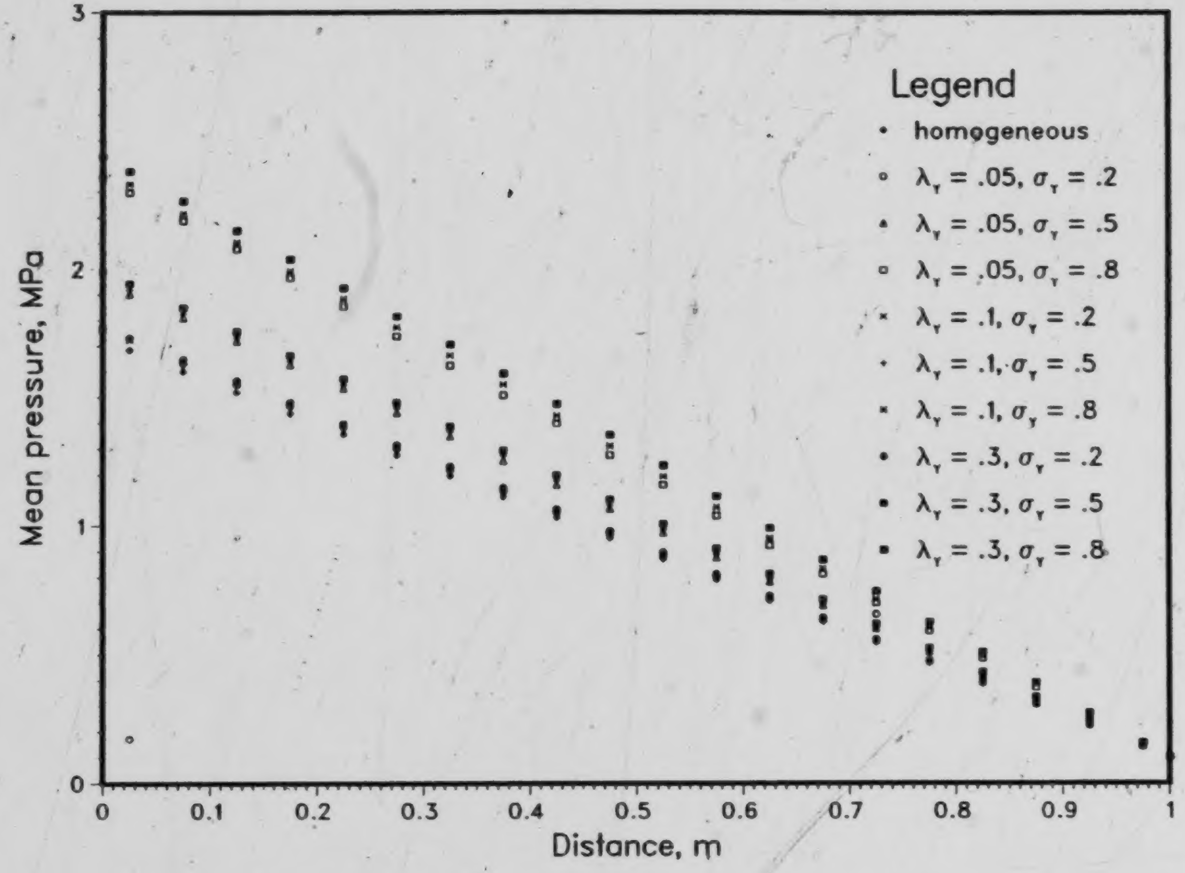


Figure VII-7 Effect of σ_Y and λ_Y' on mean pressure for single-phase flow.

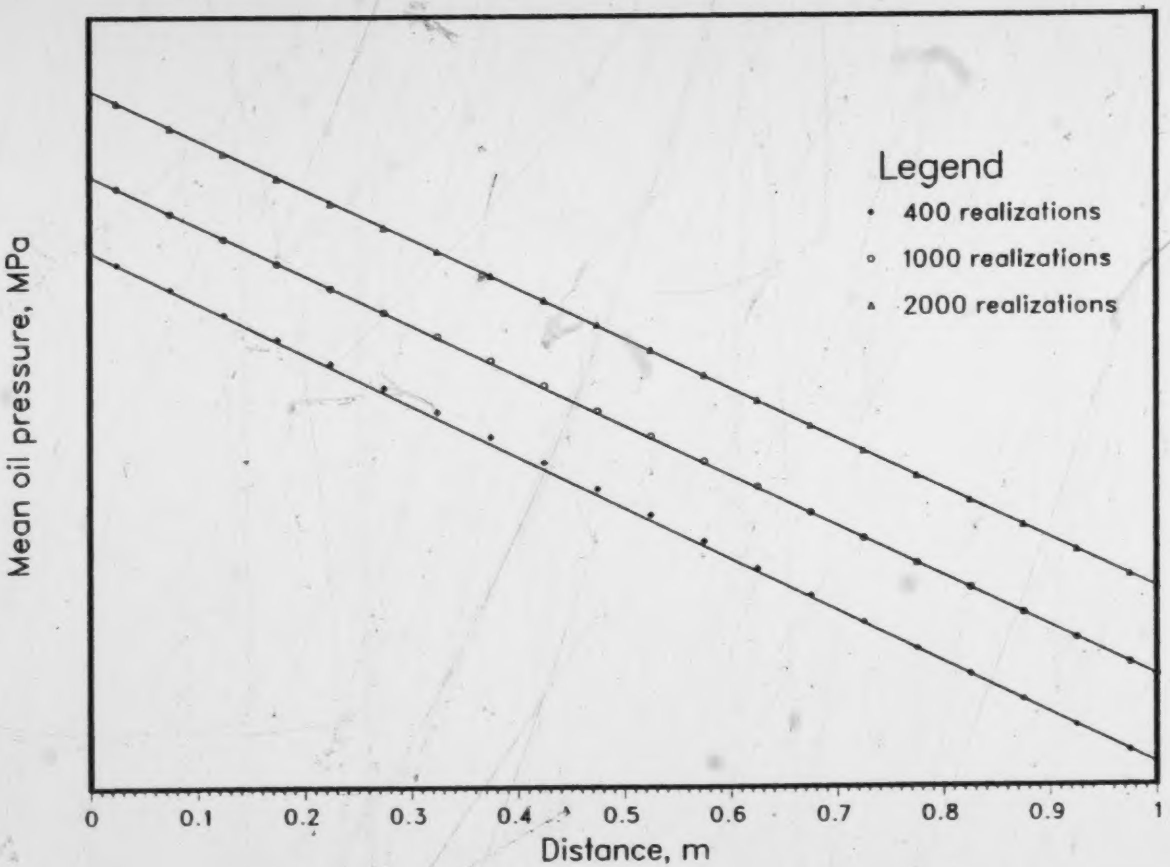


Figure VII-8 Effect of number of realizations for highly heterogeneous system ($\lambda_Y' = 0.3$, $\sigma_Y = 0.8$).

Table VII-4

Effect of number of realizations on % deviation from the straight line

Block no.	% Deviation		
	400*	1000*	2000*
1	0	0	0
2	+0.0623	+0.0213	-0.0180
3	+0.1843	+0.0069	-0.0732
4	+0.3195	-0.1020	-0.1444
5	+0.5463	-0.1543	-0.1879
6	+0.8805	-0.0707	-0.2055
7	+1.2721	+0.1018	-0.1681
8	+1.7701	+0.4697	-0.0434
9	+2.1562	+0.8081	+0.2405
10	+2.2249	+0.8895	+0.3251
11	+2.2084	+0.8574	+0.3897
12	+2.2011	+0.8462	+0.4362
13	+2.0431	+0.8541	+0.4382
14	+1.5619	+0.6541	+0.2669
15	+1.0137	+0.2629	+0.0354
16	+0.4160	-0.1540	-0.1147
17	+0.0060	-0.2007	-0.1332
18	+0.0859	+0.1480	+0.0058
19	-0.2824	+0.0923	+0.0448
20	-1.5640	-0.5560	-0.3366
21	-1.8321	-0.7088	-0.4905
22	0	0	0

* number of realizations

From this conclusion, one cautionary note should be mentioned concerning the use of Monte Carlo simulation methods for flow through porous media problems. This is that the interpretation of results obtained for high values of σ_Y and λ'_Y should be undertaken carefully because the number of realizations needed for correct results may be very high. Consequently, it may be prohibitively expensive to use Monte Carlo simulation methods for cases with high values of σ_Y and λ'_Y.

From the above discussion, it can be concluded that it is permissible to use the concept of effective permeability for both one- and two-phase flow systems. However, the standard deviation of oil pressure, caused by heterogeneity of the rock properties, has not yet been taken into consideration. Though the magnitude of the standard deviation of oil pressure does not have any effect on the general conclusion about the validity of the effective permeability concept, it will have an effect when this concept is applied to real data.

If the concept of effective permeability is valid for two-phase flow problems, what will be the effective permeability to oil and water? This question will be answered next. However, before working with that problem, let us consider the effect of σ_Y and λ'_Y on the mean water saturation. This is done because in calculation of the effective permeability to oil and water, the values of the mean water saturation are needed.

Effect of σ_Y and λ_Y' on the mean water saturation

Table VII-5 shows the maximum and minimum values of $\overline{S}_w$ along the system for each case. The greatest difference in $\overline{S}_w$ along the system (for the cases considered) is less than 0.007. Practically, this difference is very small and it may be assumed that $\overline{S}_w$ is uniform along the system with a value equal to the mean value of $\overline{S}_w$ along the system. This is shown in Table VII-6. This table reveals that there is a very small effect of σ_Y and λ_Y' on $\overline{S}_w$. The maximum difference in $\overline{S}_w$, which is between the homogeneous case and the case with $\sigma_Y = 0.8$, $\lambda_Y' = 0.3$, is less than 0.005 or 0.75% (based on the value of the homogeneous case). Therefore, it can be said that σ_Y and λ_Y' have a negligible effect on $\overline{S}_w$ for steady-state flow.

Effective permeability in two-phase flow problems

Now let us consider the effective permeability to oil. Because for one-phase flow the effective permeability can be evaluated by using Eq. (II-3), it is plausible to write the effective permeability to oil as

$$k_{eff,o} = \overline{k}_{ro}\ k_G \exp(-\tfrac{1}{2}\sigma_Y^2), \qquad \text{(VII-5)}$$

where $\overline{k}_{ro}$ is the relative permeability to oil evaluated at the mean value of $\overline{S}_w$ along the system.

To test the validity of Eq. (VII-5), the values of $k_{eff,o}$ calculated by using Eq. (VII-5) are compared with the

Table VII-5

Maximum and minimum values of $\bar{S}_w$ for steady-state flow

Case	λ'_Y	σ_Y	max. $\bar{S}_w$	min. $\bar{S}_w$	range
1	homogeneous		0.590821	0.590821	-
2	0.05	0.20	0.591881(9)*	0.590213(22)*	0.001668
3	0.05	0.50	0.594067(9)	0.589940(22)	0.004127
4	0.05	0.80	0.596928(9)	0.590397(22)	0.006531
5	0.10	0.20	0.591937(13)	0.590466(22)	0.001471
6	0.10	0.50	0.594223(13)	0.590577(22)	0.003646
7	0.10	0.80	0.597200(13)	0.591418(22)	0.005782
8	0.30	0.20	0.592057(13)	0.590858(22)	0.001199
9	0.30	0.50	0.594540(13)	0.591589(22)	0.002951
10	0.30	0.80	0.597735(13)	0.593085(22)	0.004650

* The number in the parenthesis is the number of the block where the saturation is reported.

Table VII-6

Estimated mean of $\overline{S}_v$ for steady-state flow

Case	λ_Y'	σ_Y	Est. mean of $\overline{S}_v$	%diff*
1	homogeneous		0.590821	0
2	0.05	0.20	0.591080	0.044
3	0.05	0.50	0.592075	0.212
4	0.05	0.80	0.593763	0.498
5	0.10	0.20	0.591231	0.070
6	0.10	0.50	0.592458	0.277
7	0.10	0.80	0.594385	0.603
8	0.30	0.20	0.591433	0.104
9	0.30	0.50	0.592975	0.365
10	0.30	0.80	0.595230	0.746

$$* \ \%\text{diff} = \frac{\text{mean of } \overline{S}_v - S_v \text{ of homogeneous case}}{S_v \text{ of homogeneous case}} \times 100$$

ones from the numerical experiments (Table VII-7). In the calculation of the effective permeability to oil from the results of numerical experiments, the means of inlet and outlet oil pressure are used. Column III in Table VII-7 is the effective permeability to oil obtained by using Eq. (VII-5) and $\bar{k}_{ro}$ is evaluated from $\bar{S}_w$ for the homogeneous system which is equal to 0.591 (approximately). Column IV lists the differences in percent between the effective permeabilities to oil obtained from the numerical experiment and the ones in Column III. All differences in Column IV are well below 0.3% which implies good agreement between the experimental and the theoretical results.

The relative permeabilities corresponding to the mean values of $\bar{S}_w$ for each individual case are listed in Column V. These relative permeability values are used to calculate the effective permeability to oil using Eq. (VII-5) and the k_G value from the overall estimated mean of the generated values of permeability (Column V, Table VII-7). The percent differences in the effective permeability values from Column II and Column V are shown in Column VI, Table VII-7. These percent differences are higher than those in Column IV.

One explanation for differences in Column VI being larger than those in Column IV is as follows. In the calculation of the experimental effective permeability to oil only the mean inlet and outlet pressures and the oil flow rate are used. However, as has been shown in Figure VII-5, at high values of σ_Y and λ'_Y the data points of the

Table VII-7

Comparison of effective permeability to oil from the numerical experiments and from the relationship, Eq. (VII-5)

I Case	II $k_{eff,o,exp}$	III $k^{*}_{eff,o,1}$	IV $\frac{III-II}{II}\times100$	V $k^{**}_{eff,o,2}$	VI $\frac{V-II}{II}\times100$	VII $k^{***}_{eff,o,3}$	VIII $\frac{VII-II}{II}\times100$
1	1.83166×10^{-14}	1.83166×10^{-14}	0	1.83166×10^{-14}	0	1.83166×10^{-14}	0
2	1.79118×10^{-14}	1.79139×10^{-14}	+0.0117	1.78685×10^{-14}	-0.2417	1.79084×10^{-14}	-0.0190
3	1.60645×10^{-14}	1.60736×10^{-14}	+0.0566	1.58760×10^{-14}	-1.1734	1.59657×10^{-14}	-0.6150
4	1.31782×10^{-14}	1.31810×10^{-14}	+0.0212	1.28031×10^{-14}	-2.8464	1.29193×10^{-14}	-1.9646
5	1.78468×10^{-14}	1.78495×10^{-14}	+0.0151	1.77778×10^{-14}	-0.3866	1.78818×10^{-14}	-0.1961
6	1.59230×10^{-14}	1.59296×10^{-14}	+0.0414	1.56747×10^{-14}	-1.5594	1.59057×10^{-14}	-0.1086
7	1.29921×10^{-14}	1.29627×10^{-14}	-0.2263	1.25131×10^{-14}	-3.6869	1.28393×10^{-14}	-1.1761
8	1.77569×10^{-14}	1.77640×10^{-14}	+0.0400	1.76576×10^{-14}	-0.5592	1.78463×10^{-14}	+0.5035
9	1.57125×10^{-14}	1.57427×10^{-14}	+0.1922	1.54116×10^{-14}	-1.9150	1.58244×10^{-14}	+0.7122
10	1.27205×10^{-14}	1.27493×10^{-14}	+0.2264	1.22034×10^{-14}	-4.0651	1.27310×10^{-14}	+0.0825

* Using k_G from the value of the overall estimated mean (Table VII-2) and $\overline{S}_w$ from the homogeneous case.

** Using k_G from the value of the overall estimated mean (Table VII-2) and the estimated mean of $\overline{S}_w$ for each individual case (Table VII-6).

*** Using the input k_G and the estimated mean of $\overline{S}_w$ for each individual case (Table VII-6).

plot between the mean oil pressure and distance deviate from the straight line joining the inlet and outlet pressures. This means that the experimental effective permeabilities to oil were calculated by neglecting the deviation of the mean oil pressure (from the line joining the inlet and outlet pressures) along the system. This deviation of the mean oil pressure is due to an insufficient number of realizations at this high degree of heterogeneity (see Fig. VII-8 and Table VII-4). Besides causing the mean oil pressure along the system to deviate from the straight line joining the inlet and outlet pressures, an insufficient number of realizations is believed to cause the mean values of $\bar{S}_w$ for heterogeneous cases, especially at a higher degree of heterogeneity, to be higher than that for the homogeneous case. If this is the case, the mean of $\bar{S}_w$ for all the cases considered should be very close to $\bar{S}_w$ for the homogeneous case. In fact, from Table VII-6 the mean of $\bar{S}_w$ for the case of the highest degree of heterogeneity considered is about 0.75% different from $\bar{S}_w$ for the homogeneous case. Therefore, it is probable that the mean of $\bar{S}_w$ for any heterogeneous case should be similar to $\bar{S}_w$ for the homogeneous case if a sufficient number of realizations is used. Consequently, the high percent differences in Column VI are caused by an insufficient number of realizations.

The large percentage difference in Column VI may be looked at from another point of view. As mentioned earlier, the experimental effective permeability to oil was calculated from the mean inlet and outlet pressures and the oil flow rate. This procedure might not be a good method for estimating the effective permeability to oil for a system where $\bar{S}_w$ varies from point to point along the system. If this is the case, the effective permeability to oil in Column V may represent the effective permeability to oil better than the experimental effective permeability to oil (Column II, Table VII-7). If such is the case, the percentage differences in Column VI are meaningless.

From the results for one-phase flow, the mean inlet and outlet pressures can be used to estimate the effective permeability accurately. In addition, the variation of $\bar{S}_w$ due to heterogeneity is very small. These observations lead one to prefer the former point of view. However, in both real-life applications and Monte Carlo techniques, it is unlikely that sufficient realizations are used. This is due to high cost of achieving that goal. Consequently, some discrepancy between the true effective permeability and the estimated effective permeability will always exist. Column VI in Table VII-7 shows that this discrepancy is approximately 4% for the cases with $\sigma_Y = 0.8$ and $\lambda_Y' = 0.3$. If a 4% difference is considered to be tolerable, it is possible to estimate $\bar{k}_{ro}$ from core flood tests. This will be discussed later.

Another way of calculating the values of effective permeability to oil has been tried. In this way the effective permeability to oil is calculated by using the relative permeability corresponding to the mean value of $\bar{S}_w$ for each individual case, and using the input value of k_G. The differences between effective permeabilities to oil obtained in this way and those from the numerical experiments are quite low (Column VIII, Table VII-7). In fact, the differences shown in Column VIII are quite small compared to those in Column VI. This is because, as the degree of heterogeneity increases, the estimated k_G (from the overall estimated mean of (absolute) permeability) decreases and is smaller than the input k_G. Therefore, by using the input k_G the effective permeability in Column VII, Table VII-7 will usually be higher than the experimental effective permeability in Column II. However, this will be compensated for by a high value of the mean of $\bar{S}_w$ (at high degree of heterogeneity) which causes the relative permeability to oil to be low. Hence, a lower value of the effective permeability to oil results. These two effects accidentally cause the effective permeability to oil in Column VII to be close to the experimental effective permeability to oil in Column II. Therefore, this method may not be appropriate in calculating the values of effective permeability to oil.

Effect of σ_Y and λ'_Y on the standard deviation of oil pressure

The effect of σ_Y on the standard deviation of oil pressure, s_{P_o}, is shown in Figure VII-9. It can be seen that

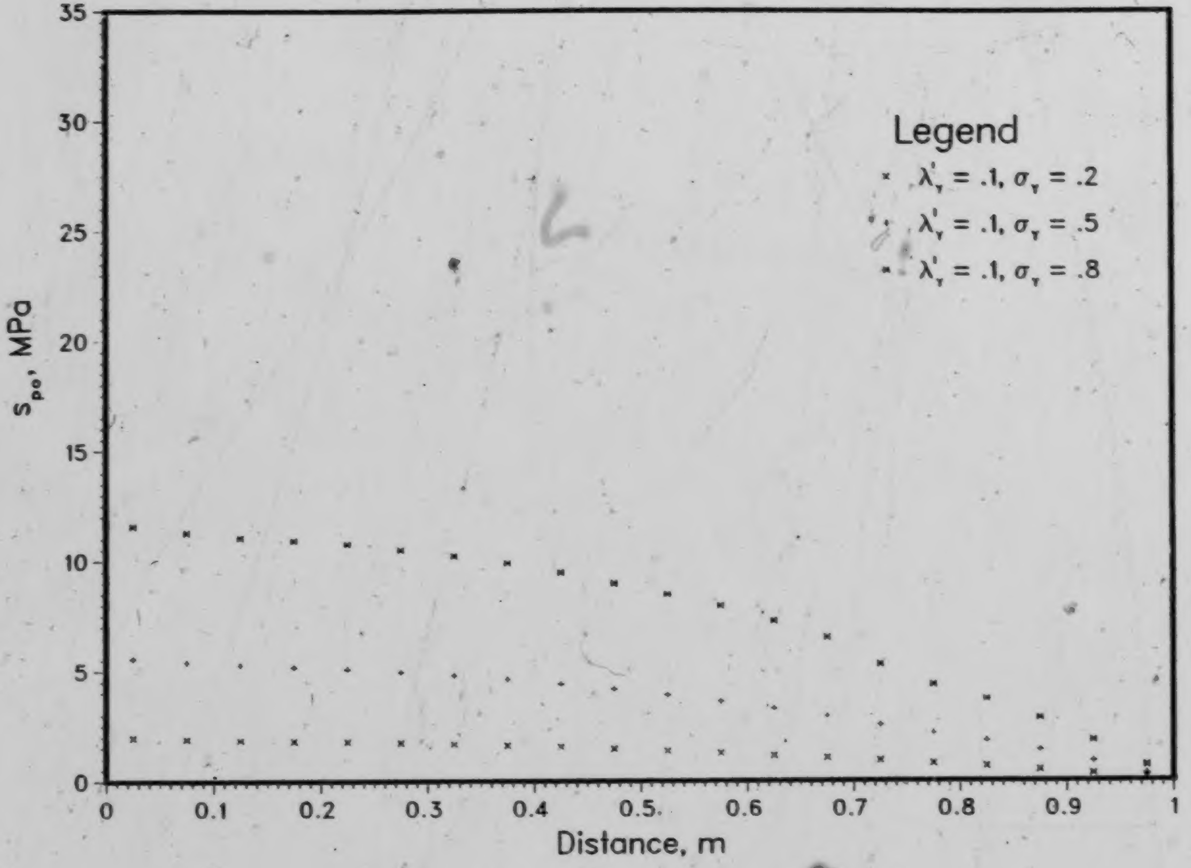


Figure VII-9 Effect of σ_γ on standard deviation of oil pressure.

as σ_Y increases, s_{p_o} increases. At the outlet end ($x = 1$ m), the value of s_{p_o} is equal to zero because the pressure at the outlet end is fixed at 0.1 MPa, i.e. there is no uncertainty in oil pressure at the outlet end. The value of s_{p_o} increases with distance from the outlet end. It seems that s_{p_o} asymptotically approaches a constant value at a location sufficiently far from the outlet end for small values of σ_Y and λ_Y'.

Figure VII-10 shows the effect of λ_Y' on s_{p_o}. Similar to the effect of σ_Y on s_{p_o}, s_{p_o} increases as λ_Y' increases and s_{p_o} increases with distance away from the outlet end. As in the case of the effect of σ_Y, the rate of increase in s_{p_o} decreases with distance away from the outlet end.

To compare the effects of σ_Y and λ_Y' on s_{p_o}, Figure VII-11 was plotted. From this figure, at $\lambda_Y' = 0.1$, as σ_Y increases 2.5 times (from 0.2 to 0.5), s_{p_o} at the inlet end increases more than 2.5 times. Similarly, as σ_Y increases 1.6 times (from 0.5 to 0.8), s_{p_o} increases more than 1.6 times. The same thing happens at $\lambda_Y' = 0.3$. Contrarily, at $\sigma_Y = 0.2$, as λ_Y' increases 3 times (from 0.1 to 0.3), s_{p_o} increases less than 3 times, and similarly for $\sigma_Y = 0.5$ and 0.8. This leads one to the conclusion that the relative effect of σ_Y on s_{p_o} is more significant than that of λ_Y'.

Let us postulate that the relationship between s_{p_o} and σ_Y and λ_Y' for an unbounded system, Eq. (IV-49), can be used for a bounded system with the addition of a coefficient, γ, as follows:

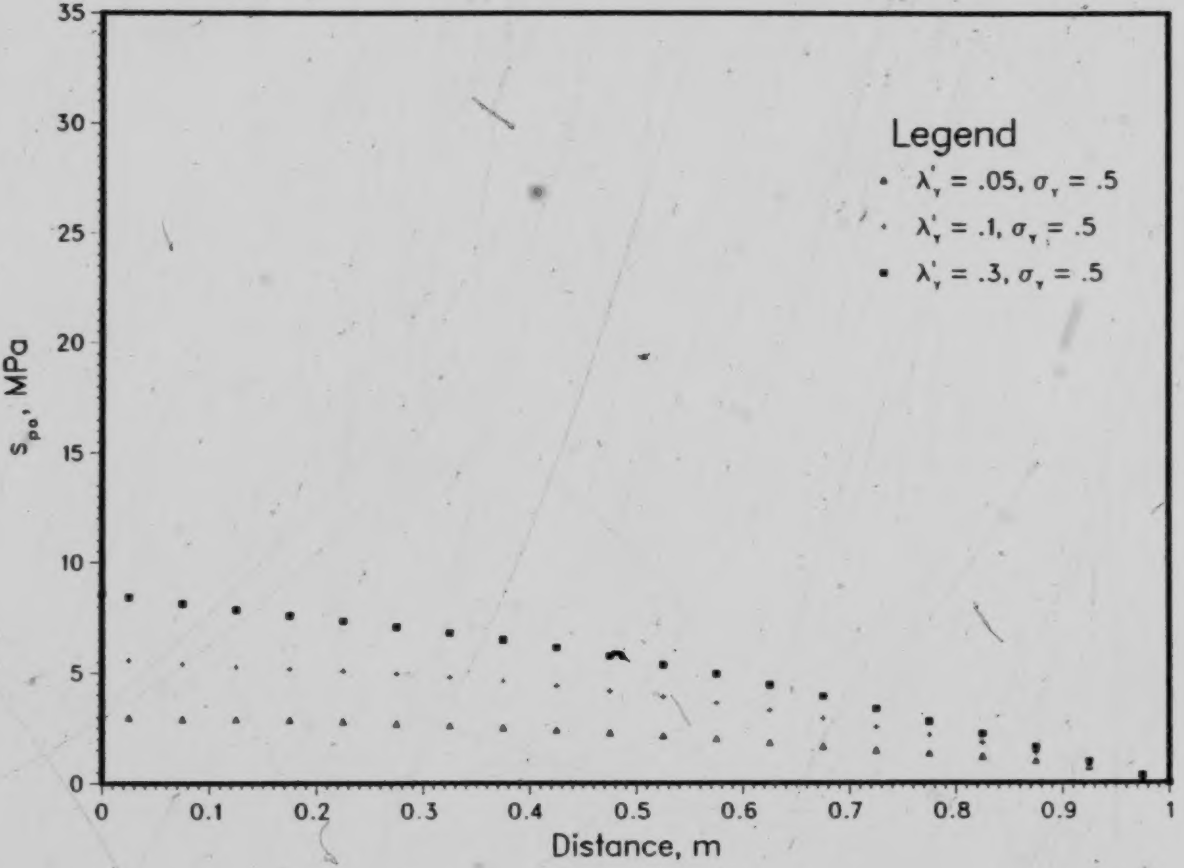


Figure VII-10 Effect of λ'_γ on standard deviation of oil pressure.

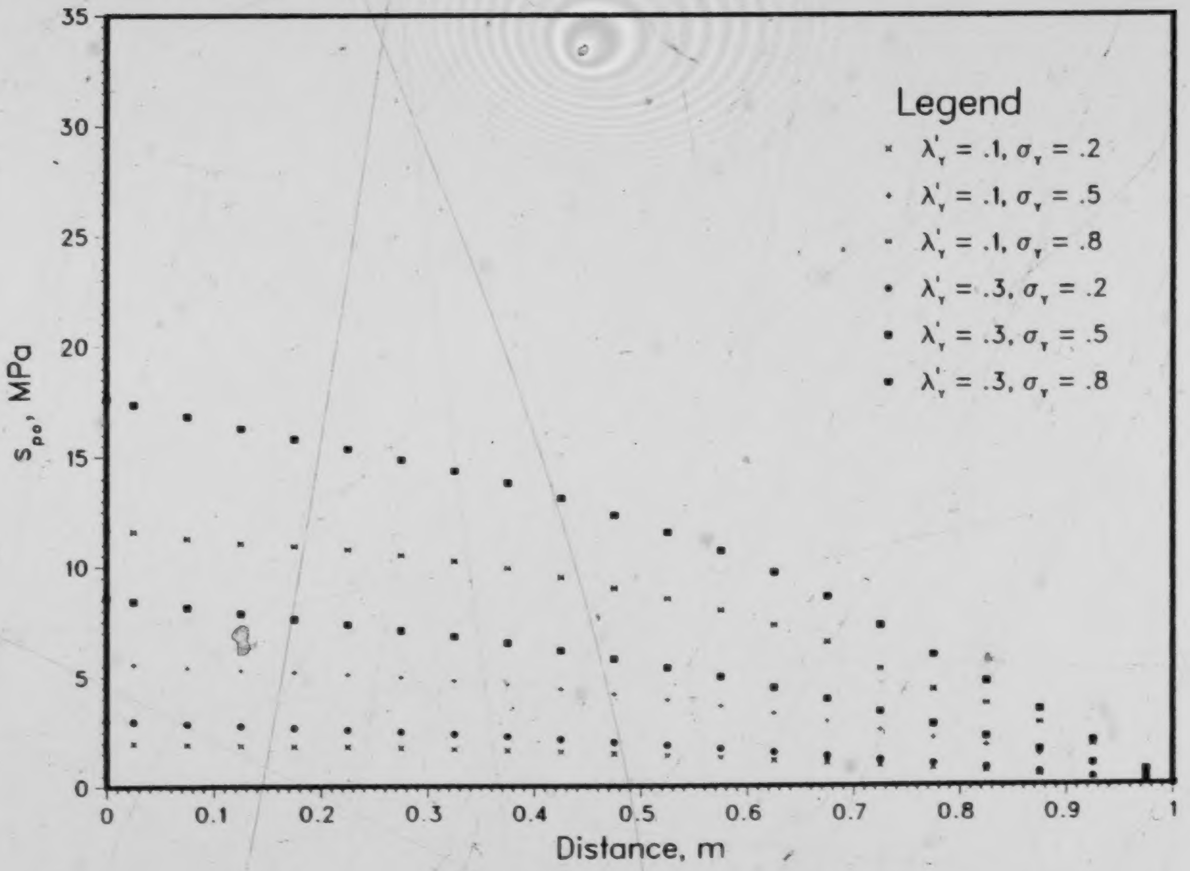


Figure VII-11 Effect of σ_γ and λ'_γ on standard deviation of oil pressure.

$$s_{p_o}^2 = \gamma\, J^2\, \sigma_Y^2\, \lambda_Y^2. \qquad \text{(VII-6)}$$

Then, with known values of s_{p_o}, J, σ_Y, and λ_Y, γ can be evaluated for each case. The values of γ calculated in this way are shown in Table VII-8 for s_{p_o} at the inlet end. At a fixed value of λ_Y', γ increases as σ_Y increases. Conversely, at fixed values of σ_Y, γ decreases as λ_Y' increases. This confirms the observation above that s_{p_o} increases at a slower rate at high values than at low values of λ_Y'. But s_{p_o} seems to increase at a faster rate at high values than at low value of σ_Y, though the difference in the rate of increase is not much. This means that though both increases in σ_Y and λ_Y' cause s_{p_o} to increase, the influence of λ_Y' becomes less as λ_Y' increases while the influence of σ_Y remains constant or slightly increases as σ_Y increases. Therefore, for a bounded system, although λ_Y' is an important parameter, σ_Y seems to be a more important one. This leads one to the conclusion that local variation of log-permeabililty seems to be a more important factor than its spatial correlation when determining s_{p_o} for a bounded domain.

The conclusion in the previous paragraph for a one-dimensional system seems to conflict with the analytical result for a three-dimensional system using spectral techniques where both σ_Y and λ_Y have equal influence on the standard deviation of oil pressure. This conflict may arise

Table VII-8

Values of coefficient γ

Case	λ'_Y	σ_Y	γ
1	homogeneous		-
2	0.05	0.2	21.88
3	0.05	0.5	23.67
4	0.05	0.8	28.60
5	0.10	0.2	19.78
6	0.10	0.5	20.30
7	0.10	0.8	22.75
8	0.30	0.2	4.96
9	0.30	0.5	5.05
10	0.30	0.8	5.46

from three possible causes. In the Monte Carlo study an one-dimensional bounded system is used while in the spectral analysis a three-dimensional unbounded system is used. This difference in dimensionality is likely to be one of the causes that make the conclusions about effect of σ_Y and λ'_Y to be different. Another likely cause of the conflict in conclusions is the difference in basic assumptions. In spectral techniques it is assumed that the perturbation of oil pressure is statistically homogeneous while in the Monte Carlo study the oil pressure is not statistically homogeneous. Finally, this conflict in conclusions may be caused partly by the fact that in the Monte Carlo study there is an insufficient number of blocks in the system and the system is too short. The number of block used in the Monte Carlo study is 20. The length of the system would be already specified when the values of λ_Y and λ'_Y are specified. Therefore, the total length of the system seems not to be a controlling parameter. In fact, Smith and Freeze[20] showed that standard deviations of pressure were approximately the same for two systems where one system was twice as long as the other, given that λ'_Y and other parameters were the same for both systems. This implies that the number of blocks in the system is not a controlling parameter either. However, the controlling parameter seems to be the number of blocks per integral scale. That is how well the system used represents the specified variation in rock properties. It is clear that the larger the number of blocks per integral scale, the better the system in representing the specified variation. Nevertheless, one cannot use too many blocks per integral scale because of the high cost in computation. The question, then, is how many blocks per integral scale should one use to obtain a system that adequately represents the specified variation. In the study by Smith and Freeze they showed that the cases with integral scale equal to around 2.7 times the block size and around 4.9 times the block size give approximately the same values of standard deviation of pressure. Systems used in this study have integral scales equal to 1, 2, and 6 times the block size. From the results of Smith and Freeze's study, it is believed that the systems which have integral scales equal to 2 and 6 times the blocks size represent the specified variation in rock preperties reasonably well.

The systems with integral scale equal to the block size were originally planned to be used as systems with uncorrelated values of log-permeability, i.e. $\lambda_Y = 0$. In fact, values of log-permeability between blocks in these systems were uncorrelated. However, it was realized later that by assigning uncorrelated values of log-permeability to blocks in a system which had block sizes comparable to the length of the system, the system would have λ_Y approximately equal to the block size. Though this is not a good representation for the specified variation, it is believed that the final results will not be far off from the ones obtained by using the system which well represents the specified variation.

From the argument above, it may be said that the different conclusions about effects of σ_Y and λ_Y' for a one-dimensional system using Monte Carlo techniques and for a three-dimensional system using spectral techniques stem from differences in dimensionality and basic assumptions.

Effect of σ_Y and λ_Y' on the standard deviation of water saturation

It can be seen from Figure VII-12 that the standard deviation of water saturation, s_{S_w}, is dependent only on σ_Y. At low values of σ_Y the data points for different λ_Y' coincide for the most part while at high values of σ_Y the data points for different λ_Y' scatter slightly. It is believed that this is not because of the effect of λ_Y' but rather because an insufficient number of realizations were used at high values of σ_Y and λ_Y'. However, the scattering of the data points at high values of σ_Y is still confined to a narrow range (maximum range is about 0.003). This suggests that s_{S_w} depends only on the local variation (σ_Y) and not at all on the spatial correlation. It can be seen from Figure VII-12 that s_{S_w} is approximately proportional to σ_Y. This is the case because s_{S_w} is likely proportional to σ_ϕ and from Eq. (V-15) it can be seen that the value of σ_ϕ used in this Monte Carlo simulation study is proportional to σ_Y. In addition, s_{S_w} is rather small (less than 0.04 at σ_Y = 0.8). Therefore, if σ_Y is small it is permissible to treat S_w as a

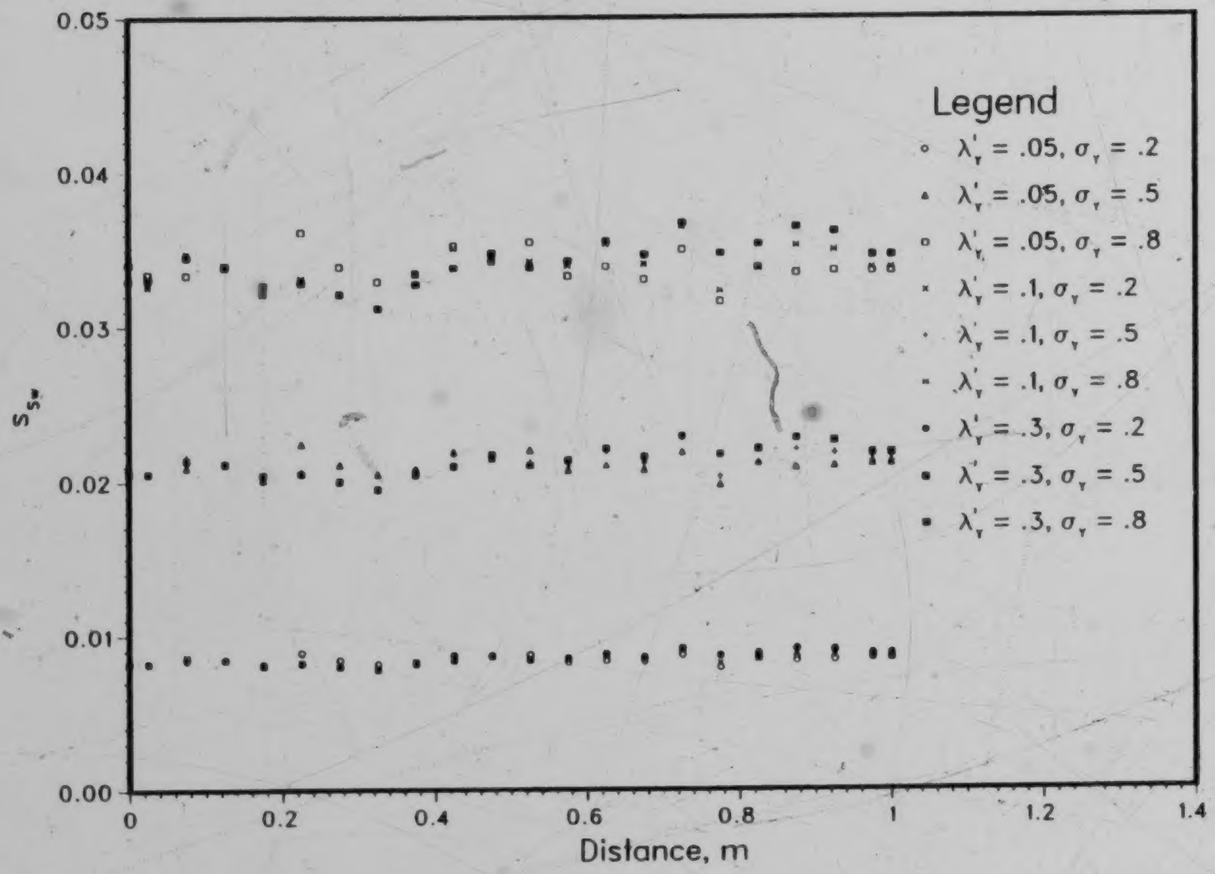


Figure VII-12 Effect of σ_Y and λ'_Y on standard deviation of water saturation.

deterministic variable. It can also be seen from Figure VII-12 is that s_{S_w} does not depend on the boundary conditions of the system.

Effect of fractional flow

All the results mentioned so far are for the case of fractional flow, f_w, equal to 0.5. It would be useful to investigate the effects of fractional flow on the means and standard deviation of oil pressure and saturation at fixed values of σ_Y and λ'_Y. Figure VII-13 shows a plot of $\bar{p}_o$ versus x. The highest pressure drop is at $f_w = 0.4$. This is because of the input functional form of relative permeability and the input value of mobility ratio. The interesting result is that the magnitude of s_{p_o} varies in the same direction as the magnitude of the pressure gradient (Figure VII-14). This confirms the validity of Eq. (IV-49) that s_{p_o} varies with the mean pressure gradient. In other words, s_{p_o} varies as $\bar{S}_w$ varies, other things being kept constant, where $\bar{S}_w$ controls f_w and $\bar{k}_{ro}$.

As expected, $\bar{S}_w$ increases as f_w increases (Figure VII-15). The variation of $\bar{S}_w$ along the system is quite small. The effect of f_w on s_{S_w} is shown in Figure VII-16. It can be seen that s_{S_w} varies in a direction opposite to that of $\bar{S}_w$. This was puzzling at first. However, from the formulation of the problem, S_{wirr} is designed to be closely related to the heterogeneity of the rock properties. At lower values of $\bar{S}_w$, S_{wirr} will contribute to the total

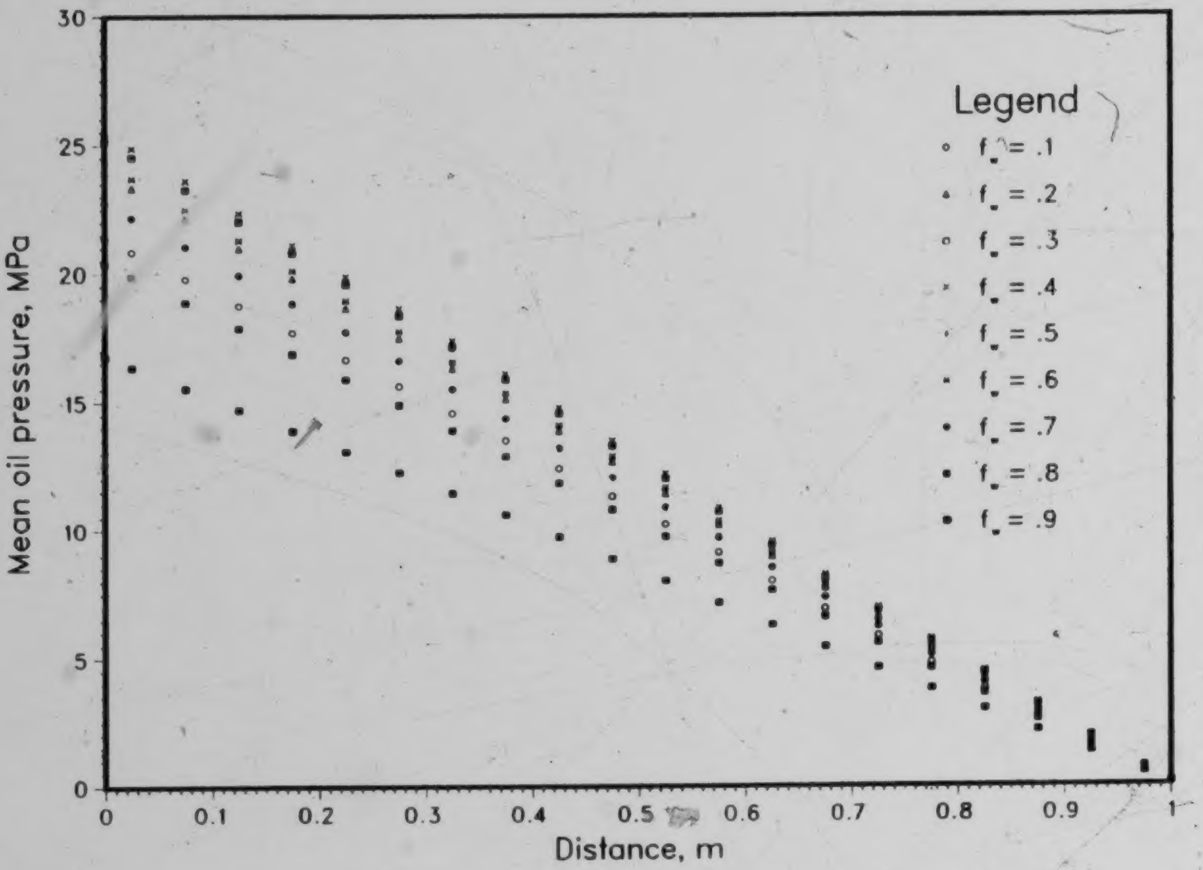


Figure VII-13 Effect of f_w on $\overline{p}_o$, $\sigma_Y = 0.5$, $\lambda'_Y = 0.1$.

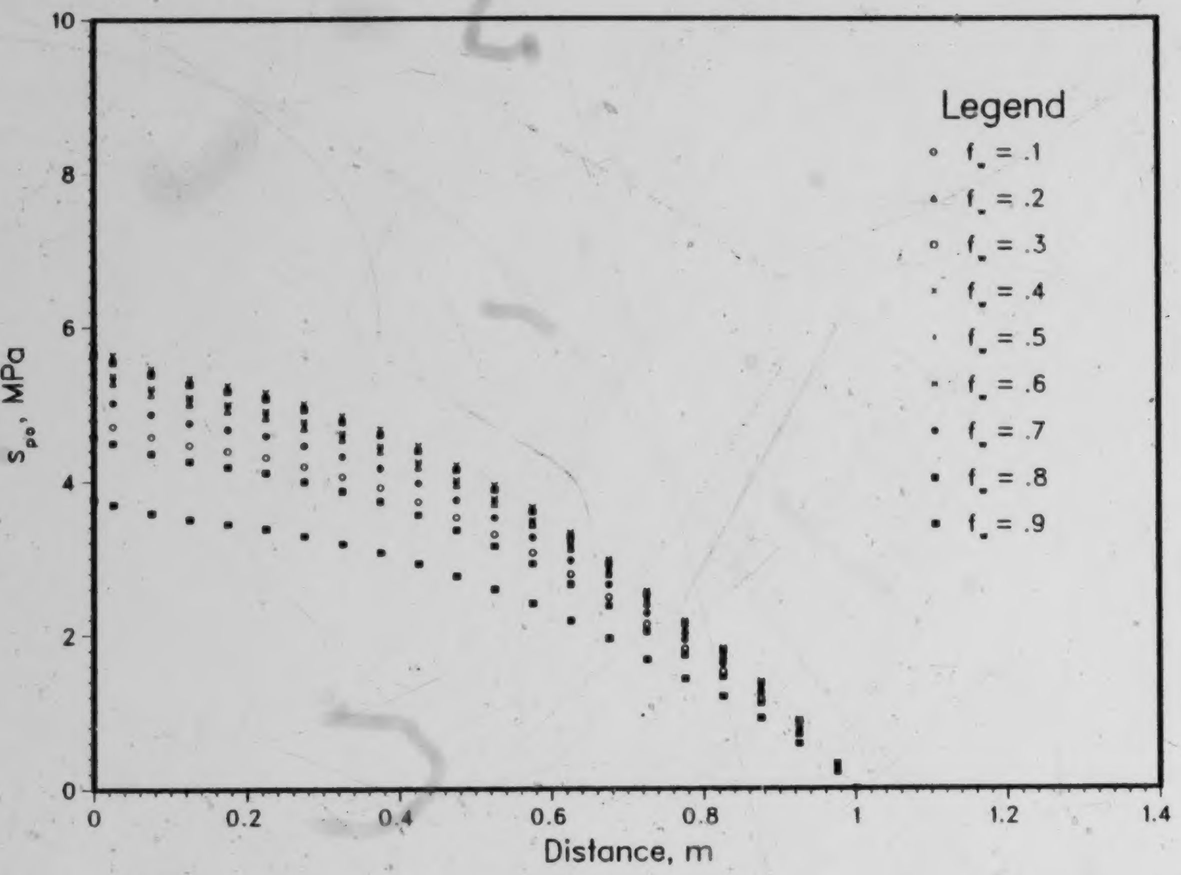


Figure VII-14 Effect of f_w on s_{p_o}, $\sigma_Y = 0.5$, $\lambda_Y' = 0.1$.

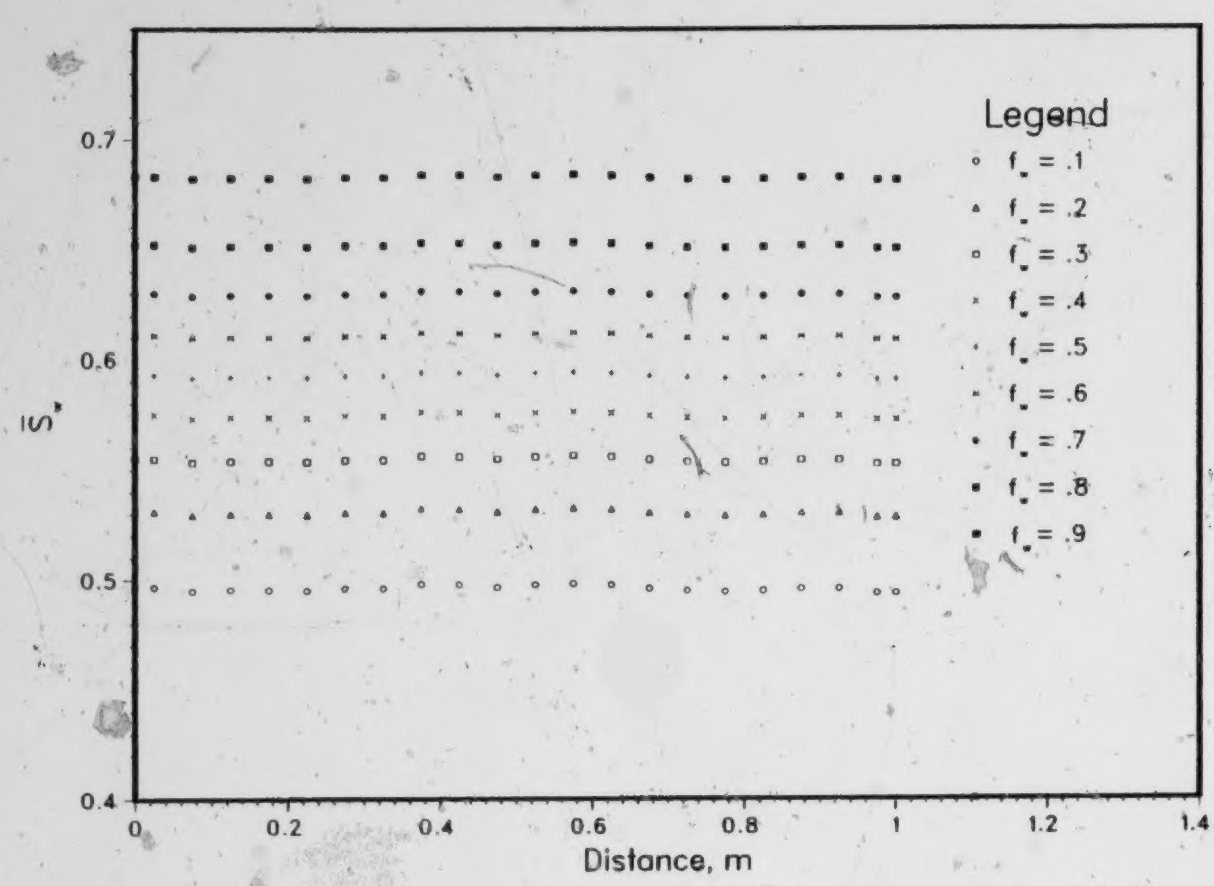


Figure VII-15 Effect of f_w on $\bar{S}_w$ [illegible] $\sigma_r = 0.5$, $\lambda_r' = 0.1$ ($\mu_o/\mu_w = 5$, $P_c = 0$).

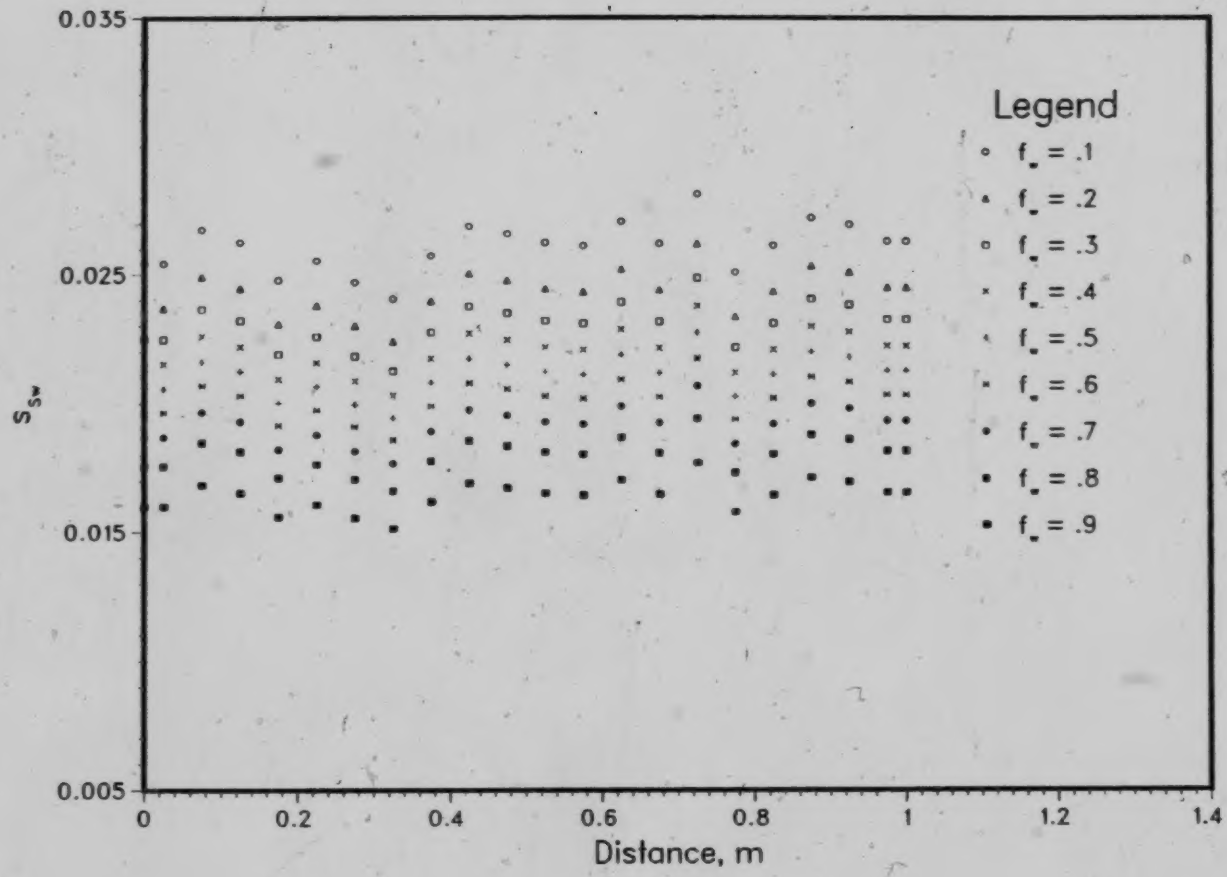


Figure VII-16 Effect of f_w on s_{s_w}, $\sigma_Y = 0.5$, $\lambda_Y' = 0.1$ ($\mu_o/\mu_w = 5$, $P_c = 0$).

value of $\overline{S}_w$ more than at higher values of $\overline{S}_w$. Therefore, the variation of the rock properties will be more pronounced at low values of $\overline{S}_w$ than at high values of $\overline{S}_w$.

Effect of mobility ratio

A viscosity ratio, μ_o/μ_w, of 5 has been used for the results discussed so far. This value of the viscosity ratio corresponds to an end-point mobility ratio, $k_{wor}\mu_o/k_{oiw}\mu_w$, of 1.6 which is unfavorable. The results for the favorable cases (viscosity ratio = 1.0 and end-point mobility ratio = 0.3) were obtained for all cases of the same set of input σ_Y and λ_Y' as for the unfavorable cases. It can be seen from Figures VII-17 to VII-19 that σ_Y and λ_Y' have similar effects (as for the unfavorable cases) on $\overline{p}_o$, s_{p_o}, and s_{S_w}. The only exception is that by using the same total rate, the magnitudes of $\overline{p}_o$ and s_{p_o} are smaller for the favorable cases and $\overline{S}_w$ is greater, while s_{S_w} is slightly smaller for the favorable cases.

Effect of capillary pressure

Zero capillary pressure has been used for the above results. When capillary pressure was taken into consideration, it turned out that there were virtually no changes in the values of s_{p_o}, $\overline{S}_w$, and s_{S_w}. Therefore, capillary pressure has no effect on s_{p_o} and s_{S_w} for steady-state flow.

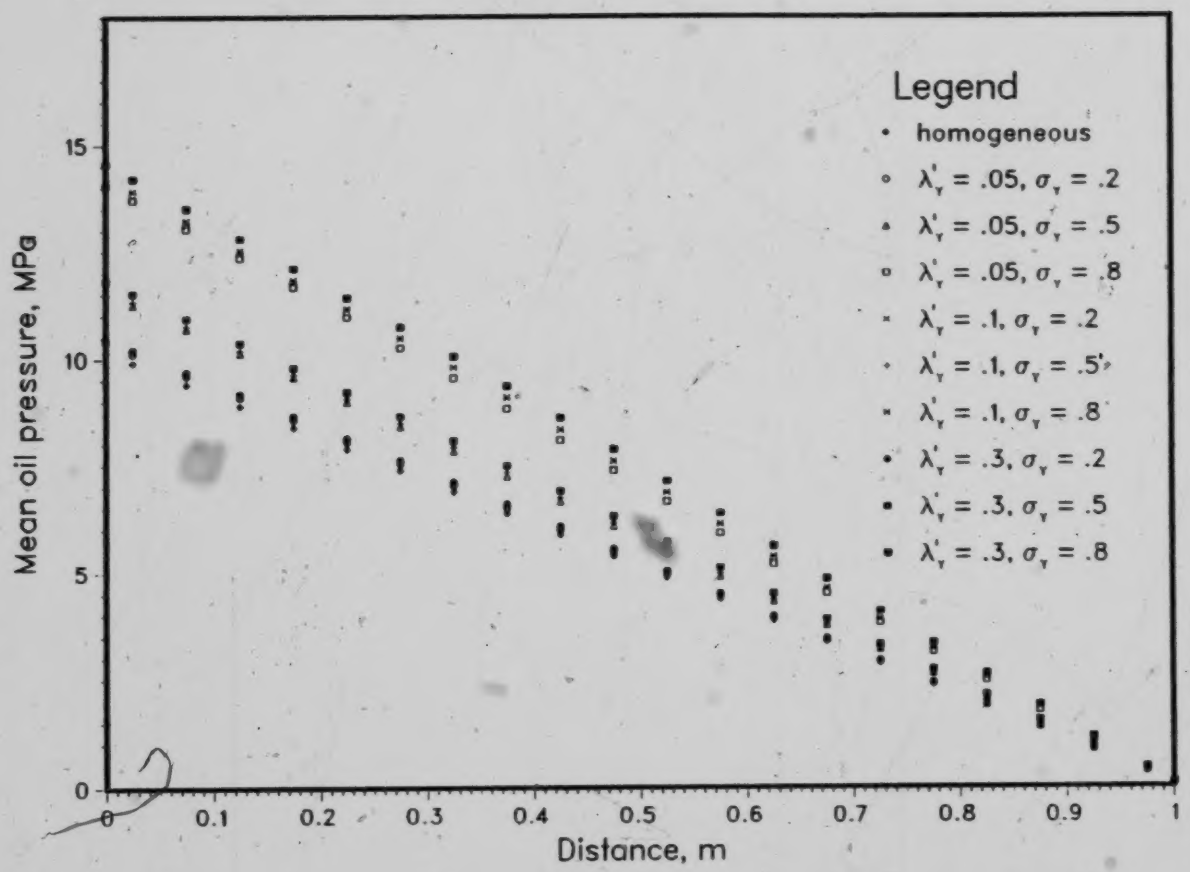


Figure VII-17 Effect of σ_γ and λ'_γ on mean of oil pressure (favourable cases).

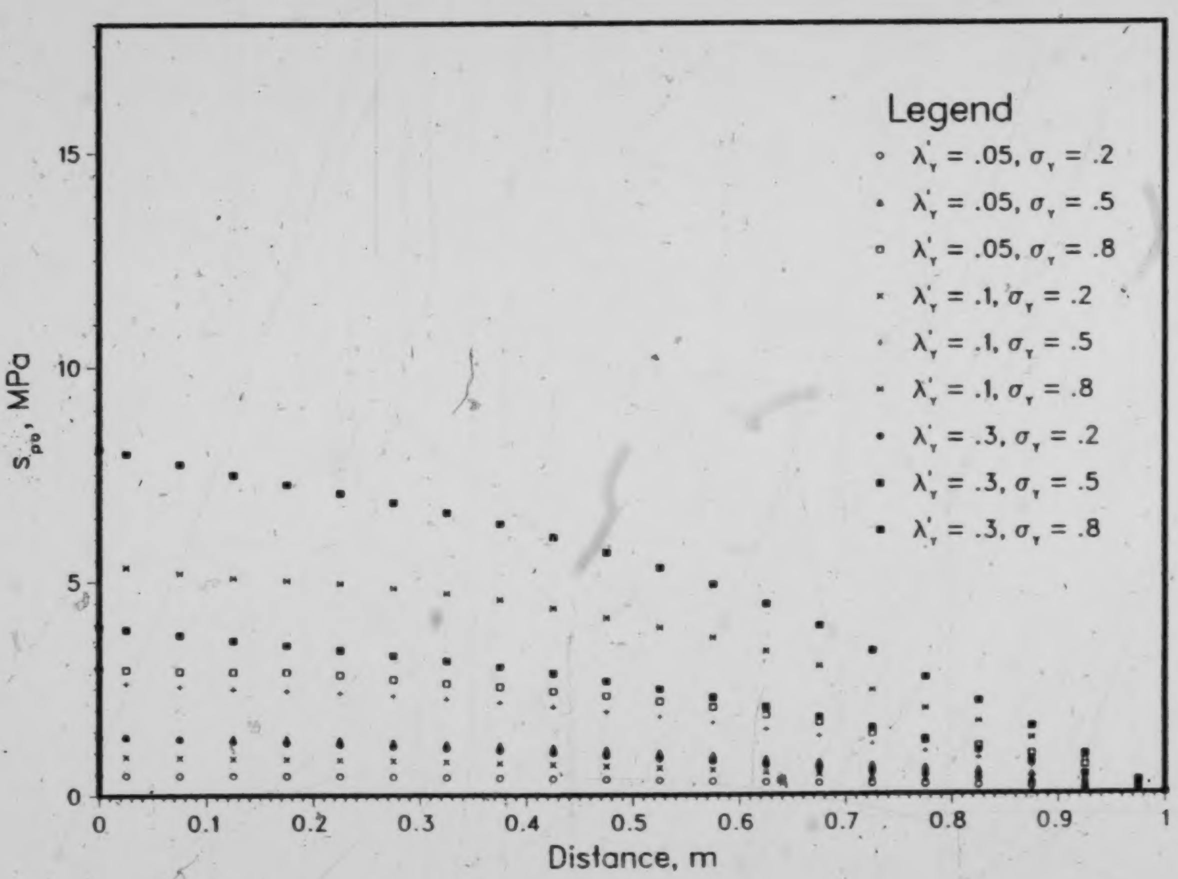


Figure VII-18 Effect of σ_γ and λ'_γ on standard deviation of oil pressure (favourable cases).

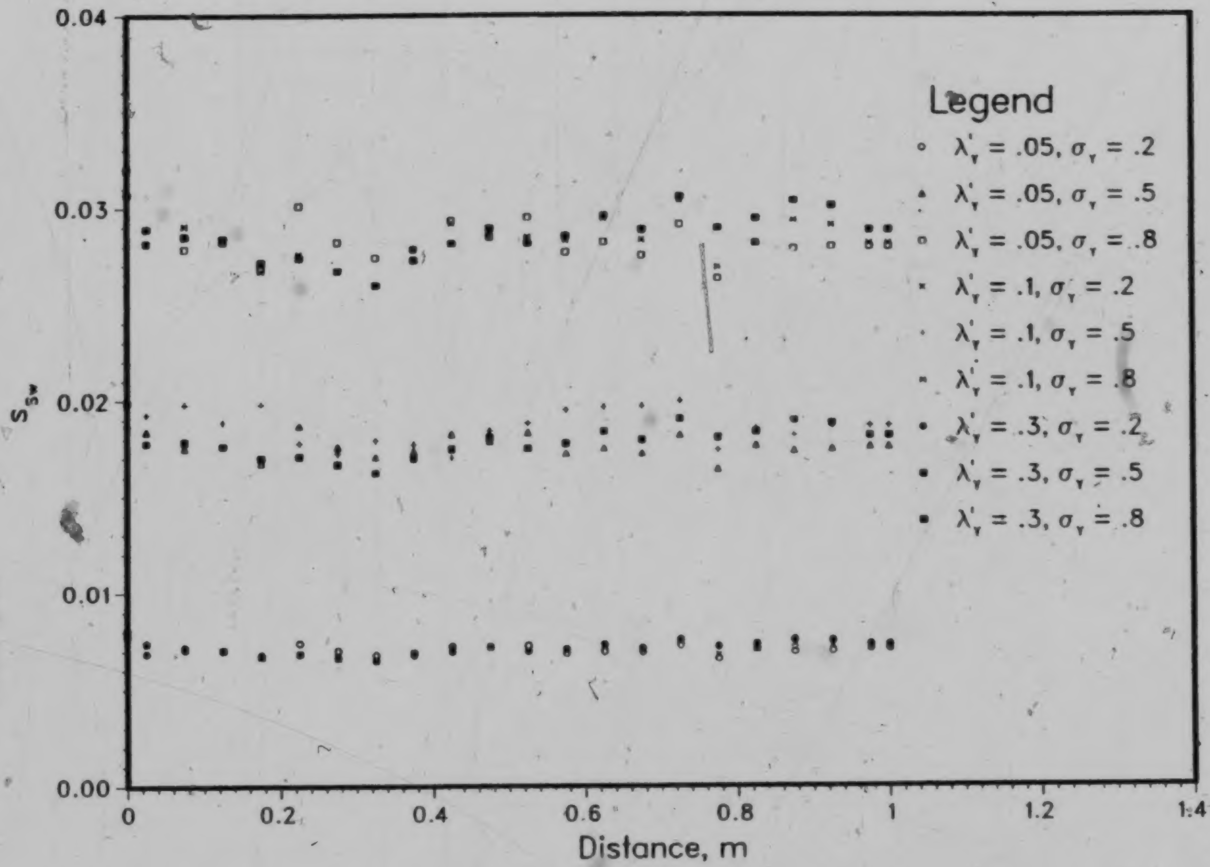


Figure VII-19 Effect of σ_Y and λ'_Y on standard deviation of water saturation (favourable cases).

Effect on plot of fractional flow versus mean of water saturation

The values of f_w are plotted against the means of $\bar{S}_w$ for the homogeneous case (solid curve) and the case with $\sigma_Y = 0.5$, and $\lambda_Y' = 0.1$ (circle) in Figure VII-20. For all practical purposes, it may be said that the two cases give the same f_w versus $\bar{S}_w$ curve. It is believed that the case with $\sigma_Y = 0.8$ and $\lambda_Y' = 0.3$ (which is the case of the highest degree of heterogeneity considered in this study) will also give approximately the same curve as the previous two cases because at $f_w = 0.5$, the difference in the mean of $\bar{S}_w$ of the homogeneous case and this case is only 0.004. Therefore, the effect of σ_Y and λ_Y' on the plot of f_w versus the mean of $\bar{S}_w$ is insignificant, at least within the degree of heterogeneity considered here.

C. Constant pressure at both ends

This type of boundary condition has no uncertainty in pressure at both ends. However, the pressures inside the system are still uncertain. In addition, the flow rate is now a random variable and its value depends on the degree of heterogeneity of the rock properties. It will be shown later that this type of boundary condition is not appropriate for estimating the effective permeability of a serial system. In the following, the effect of heterogeneity of rock properties on the resulting pressure and saturation will be discussed in detail.

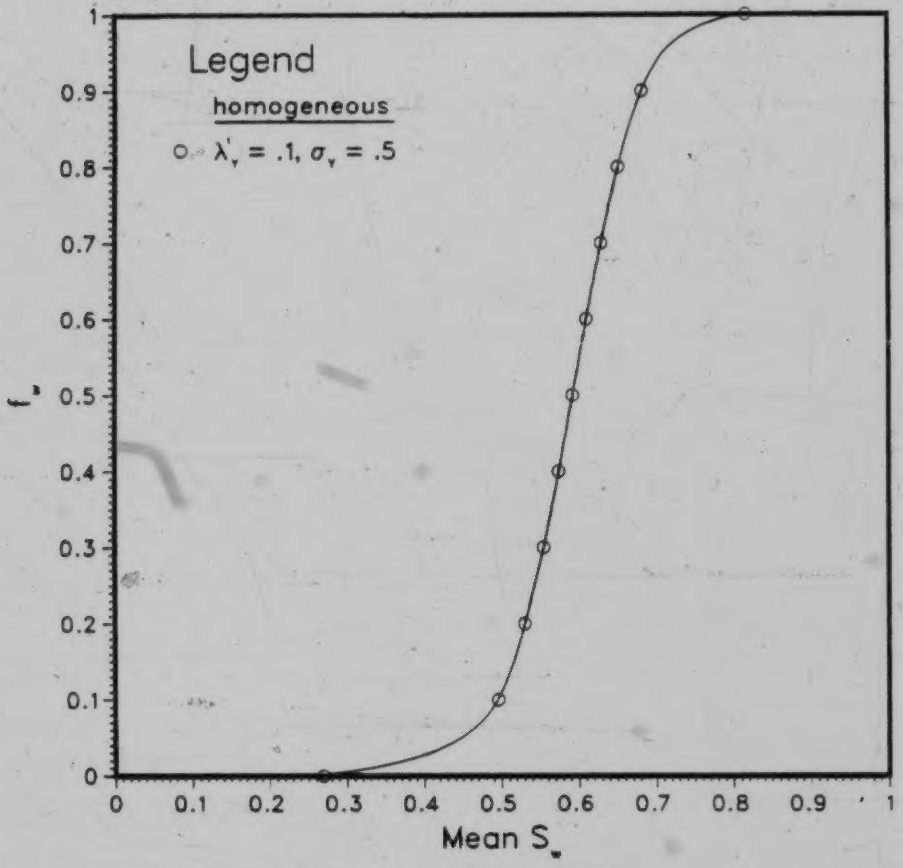


Figure VII-20 Effect of σ_γ and λ'_γ on fractional flow curve.

Effect of σ_Y and λ_Y' on the mean oil pressure

It is shown (Figure VII-21) that both σ_Y and λ_Y' have almost no effect on $\bar{p}_o$ for this type of boundary condition. At high values of σ_Y and λ_Y', a few data points deviate from the straight line joining the inlet and outlet pressures. These deviations are quite small and may be neglected. The mean oil pressure is unaffected by rock heterogeneity because of the boundary condition of fixed pressure at both ends.

Effect of σ_Y and λ_Y' on the mean flow rate

Although $\bar{p}_o$ is practically independent of both σ_Y and λ_Y', the mean oil flow rate, $\bar{q}_o$, is dependent to a large extent on these two parameters (Figure VII-22). The value of $\bar{q}_o$ decreases as σ_Y increases and λ_Y' decreases. When σ_Y increases, there is more variation in the values of permeability in each realization. This will cause a low permeability block to be a part of most realizations. Because of the nature of flow in serial systems, one low permeability block can cause the oil flow rate to decrease considerably. However, at high values of λ_Y', each realization will consist of blocks which have permeability values close to one another. Hence, realizations that consist of a set of high permeability blocks will have high flow rates while realizations which consist of a set of low permeability blocks will have low flow rates. When all the flow rates are (arithmetically) averaged, the high flow rate

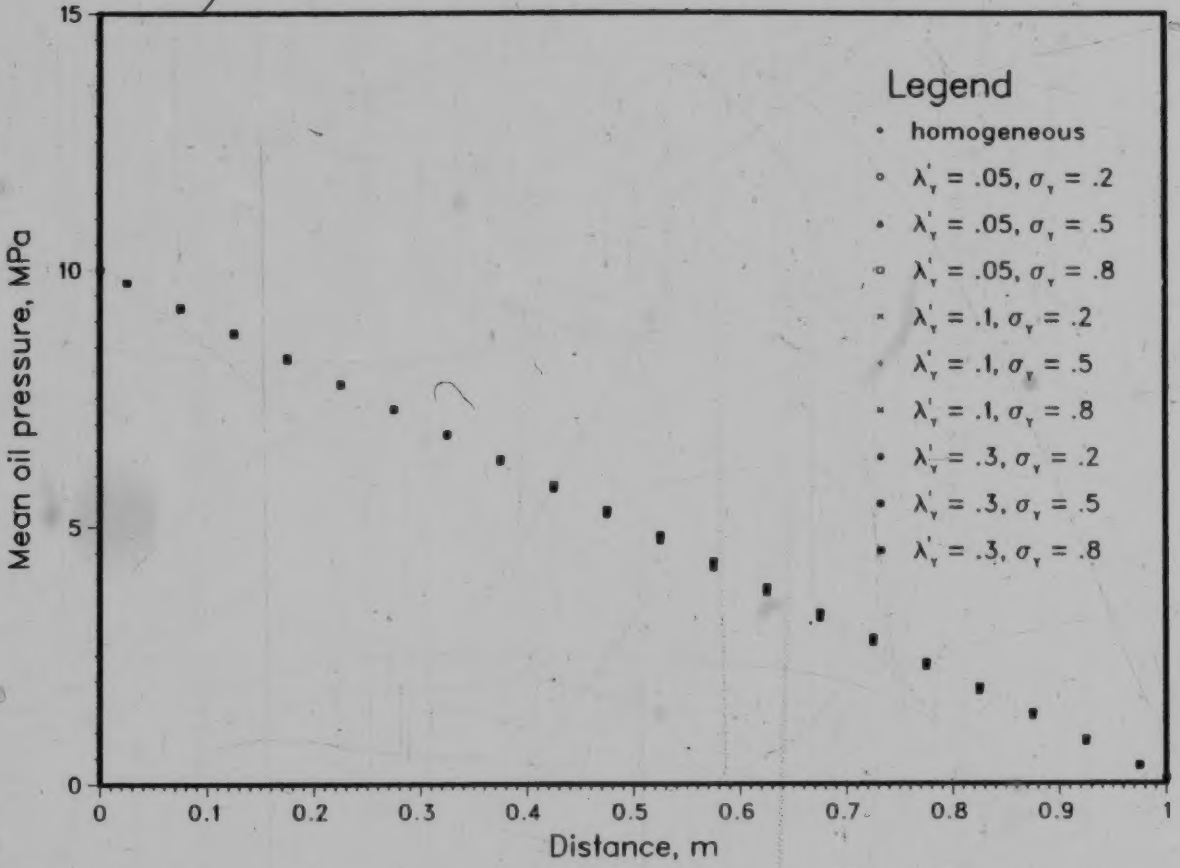


Figure VII-21 Effect of σ_Y and λ'_Y on mean of oil pressure (constant pressure at both ends).

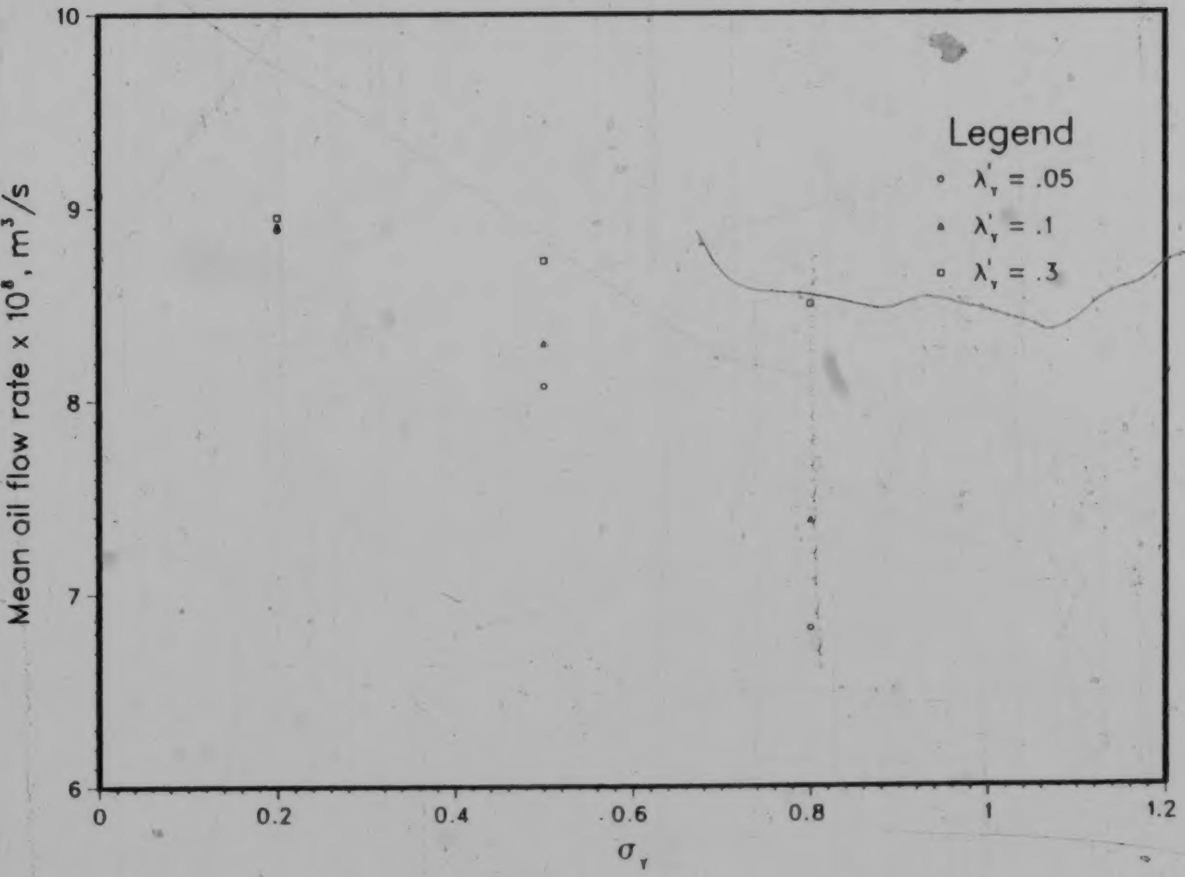


Figure VII-22 Effect of σ_Y and λ'_Y on mean of oil flow rate.

values will dominate and the mean oil flow rate will be high for high values of λ'_Y.

Now let us look at the effective permeability to oil for this type of boundary condition and compare it to the one obtained from the expression in Eq. (VII-5). Table VII-9 shows that the differences between the effective permeability to oil from the numerical experiment and the effective permeability to oil from the analytical expression are high for cases having high values of σ_Y and λ'_Y. In addition, the effective permeabilities to oil from the analytical expression are consistently lower. These large differences necessitate an explanation to clarify the matter. Either the analytical expression for effective permeability is not right for the system considered (serial system) or the boundary conditions for this type of system are not appropriate to evaluate the effective permeability for a serial system.

This doubt can be clarified by the concept which is used in Appendix A to derive the effective permeability for parallel and serial systems. There it is postulated that, for a serial system, treating the flow rate as a deterministic variable is more appropriate than treating it as a random variable. Using the kind of boundary conditions considered here the flow rate is treated as a random variable. Therefore, it is not appropriate to use fixed inlet and outlet pressures to estimate the effective permeability of a serial system. It is also postulated in

Table VII-9

Effective permeability to oil for cases with constant pressure at both ends

Case	λ_Y'	σ_Y	$k_{eff,o,num}$	$k^*_{eff,o,ana}$	%diff.
1	homogeneous		1.83166×10^{-14}	1.83166×10^{-14}	0
2	0.05	0.20	1.79513×10^{-14}	1.79139×10^{-14}	-0.2083
3	0.05	0.50	1.63093×10^{-14}	1.60736×10^{-14}	-1.4452
4	0.05	0.80	1.37758×10^{-14}	1.31810×10^{-14}	-4.3177
5	0.10	0.20	1.79901×10^{-14}	1.78495×10^{-14}	-0.7815
6	0.10	0.50	1.67588×10^{-14}	1.59296×10^{-14}	-4.9478
7	0.10	0.80	1.49062×10^{-14}	1.29627×10^{-14}	-13.0382
8	0.30	0.20	1.80821×10^{-14}	1.77640×10^{-14}	-1.7592
9	0.30	0.50	1.76245×10^{-14}	1.57427×10^{-14}	-10.6772
10	0.30	0.80	1.71655×10^{-14}	1.27493×10^{-14}	-25.7272

* evaluated by using k_{ro} obtained from $\bar{S}_w$ of the homogeneous case and k_G is obtained from the overall mean of log-permeability for each case.

Appendix A that for a parallel system the pressure gradient should be treated as a deterministic variable while the flow rate should be treated as a random variable. Therefore, this type of boundary condition should be used to estimate the effective permeability of a parallel system.

These conclusions lead to questions concerning two- and three-dimensional systems about which variable should be treated as a random variable and/or whether both flow rate and pressure gradient should be treated as random variables. (It should be noted that permeability and, possibly, porosity have already been treated as random variables.) Because two- and three-dimensional systems are intermediate between (one-dimensionally) serial and parallel systems, it is likely that both pressure gradient and flow rate have to be treated, along with permeability, as random variables for two- and three-dimensional systems. If this is the case, either probability distributions for pressure gradient or flow rate will be needed when analyzing the flow equations for two- and three-dimensional systems by Monte Carlo techniques. However, as can be seen from Table VII-9, at low values of σ_Y and λ_Y', the differences between the effective permeability from the numerical experiment and that from the analytical expression are not too large. Consequently, it is possible to say that at low values of σ_Y and λ_Y', either the pressure gradient or the flow rate can be treated as a deterministic variable for n-dimensional systems, where $n = 1, 2$, or 3.

In fact, the observation about the estimation of effective permeability is in the same direction as the results of Smith and Freeze.[21] They concluded from their results that both the nature of the spatial heterogeneities in permeability and the flow system operating within the flow domain must be considered when evaluating the concept of an effective permeability for a heterogeneous system.

Because the estimation of effective permeability is affected by whether the pressure gradient or the flow rate or both are treated as random variables, it is quite probable that this same problem may affect the estimation of the standard deviation of pressure and (for two-phase flow) saturation. If this is the case, care must be taken in setting up the problem for statistical analysis of flow through a porous medium in a one-, two-, or three-dimensional system. Treating the pressure gradient as a constant, like in a deterministic formulation, may not be appropriate for the statistical formulation of the problem unless it is the condition that is prevailing in the field under consideration. In addition, when comparing s_{p_o} for a one-dimensional system to that for a two- or three-dimensional system, the correct formulation for each individual system must be used. When Smith and Freeze[21] compared the standard deviation of pressure for one- and two-dimensional systems, they used a constant pressure gradient for both systems (or, in other words, they treated the pressure gradient as a deterministic variable). Their reduction in the standard deviation of pressure values from a one- to a two-dimensional system is much less than that of Bakr *et al.*[18] (from one- to three-dimensional systems). (It should be noted that the reduction in standard deviation of pressure from a two- to three-dimensional system is small.[6,8,23]) One of the reasons underlying the difference in reduction of standard deviation of pressure may be that while Smith and Freeze[21] treated the pressure gradient as a deterministic variable, Bakr *et al.*[18] treated flow rate as a deterministic variable.

Effect of σ_Y and λ'_Y on the mean of water saturation

As in the case of constant rate and constant pressure at the outlet end, the effect of σ_Y and λ'_Y on $\bar{S}_w$ is very small. Mean, maximum, and minimum values of $\bar{S}_w$ along the system are shown in Table VII-10. In fact, these values are very similar to those for the case with constant rate and contant pressure at the outlet end. Only after the third decimal place are there differences in the figures of the mean of $\bar{S}_w$. The maximum difference along the system for all cases is less than 0.01 which is very small. Therefore, the effect of σ_Y and λ'_Y on $\bar{S}_w$ can be neglected.

Effect of σ_Y and λ'_Y on the standard deviation of oil pressure

Both σ_Y and λ'_Y have a considerable effect on s_{p_o}. The highest value of s_{p_o} is at the middle of the system because

Table VII-10

Mean, maximum and minimum values of $\bar{S}_v$ along the system (steady-state flow, constant pressure at both ends)

Case	λ_r'	σ_r	mean $\bar{S}_v$	max. $\bar{S}_v$	min. $\bar{S}_v$	range
1	homogeneous		0.590821	-	-	-
2	0.05	0.20	0.591058	0.591881(9)*	0.590213(22)*	0.001668
3	0.05	0.50	0.592023	0.594067(9)	0.589940(22)	0.004127
4	0.05	0.80	0.593680	0.596928(9)	0.590397(22)	0.006531
5	0.10	0.20	0.591207	0.591937(13)	0.590466(22)	0.001471
6	0.10	0.50	0.592398	0.594223(13)	0.590577(22)	0.003646
7	0.10	0.80	0.594282	0.597200(13)	0.591418(22)	0.005782
8	0.30	0.20	0.591408	0.592057(13)	0.590858(22)	0.001199
9	0.30	0.50	0.592915	0.594540(13)	0.591589(22)	0.002951
10	0.30	0.80	0.595123	0.597735(13)	0.593085(22)	0.004650

* The number in the parenthesis is the number of the block where the saturation is reported.

the pressures at both ends are known with certainty. The value of s_{p_o} increases as σ_Y increases (Figure VII-23) and as λ'_Y increases (Figure VII-24). It can be seen that σ_Y has a larger effect on s_{p_o} than does λ'_Y (Figure VII-25). This leads one to a similar conclusion, drawn earlier, that s_{p_o} is dependent more on local variation than on spatial correlation. Another interesting obervation is that at fixed σ_Y, the rate of increase in s_{p_o} decreases as λ'_Y increases. On the other hand, s_{p_o} increases steadily as σ_Y increases. This further confirms the significance of local variation.

Effect of σ_Y and λ'_Y on the standard deviation of oil flow rate

The effect of σ_Y and λ'_Y on the standard deviation of the oil flow rate, s_{q_o}, is significant (Figure VII-26). The high rate of increase of s_{q_o} as σ_Y and λ'_Y increase is probably due to the nature of the flow through a serial system. As mentioned before, there are large differences in flow rate from realization to realization at high values of σ_Y and λ'_Y. This causes a large standard deviation of the oil flow rate. For a parallel system the situation would be different. It is believed that for a parallel system, the dependence of s_{q_o} on σ_Y and λ'_Y will be much smaller than that of the serial system. This is because in the parallel system flow can divert away from blocks of low permeability. The flow rate for each realization will not be as much different as that of the serial system. More about flow in a parallel

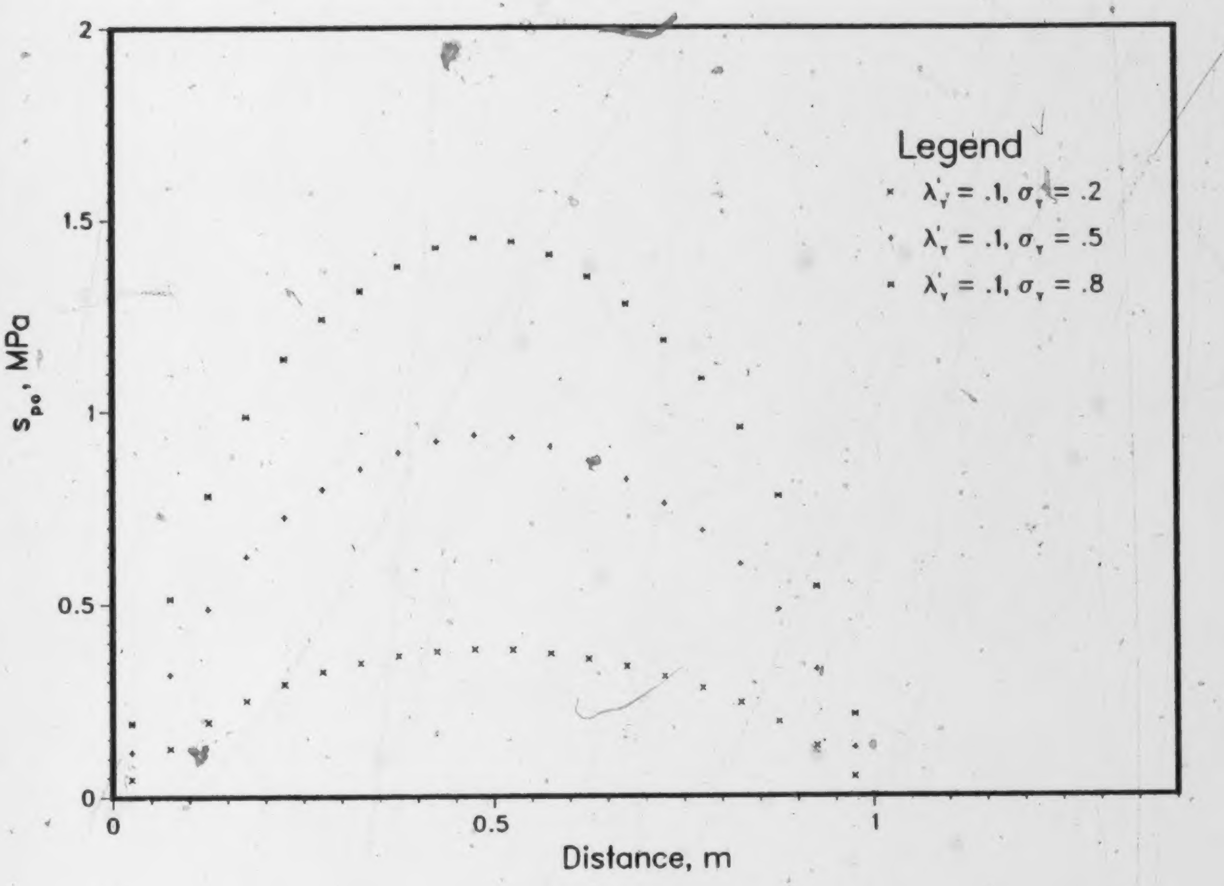


Figure VII-23 Effect of σ_γ on standard deviation of oil pressure (constant pressure at both ends).

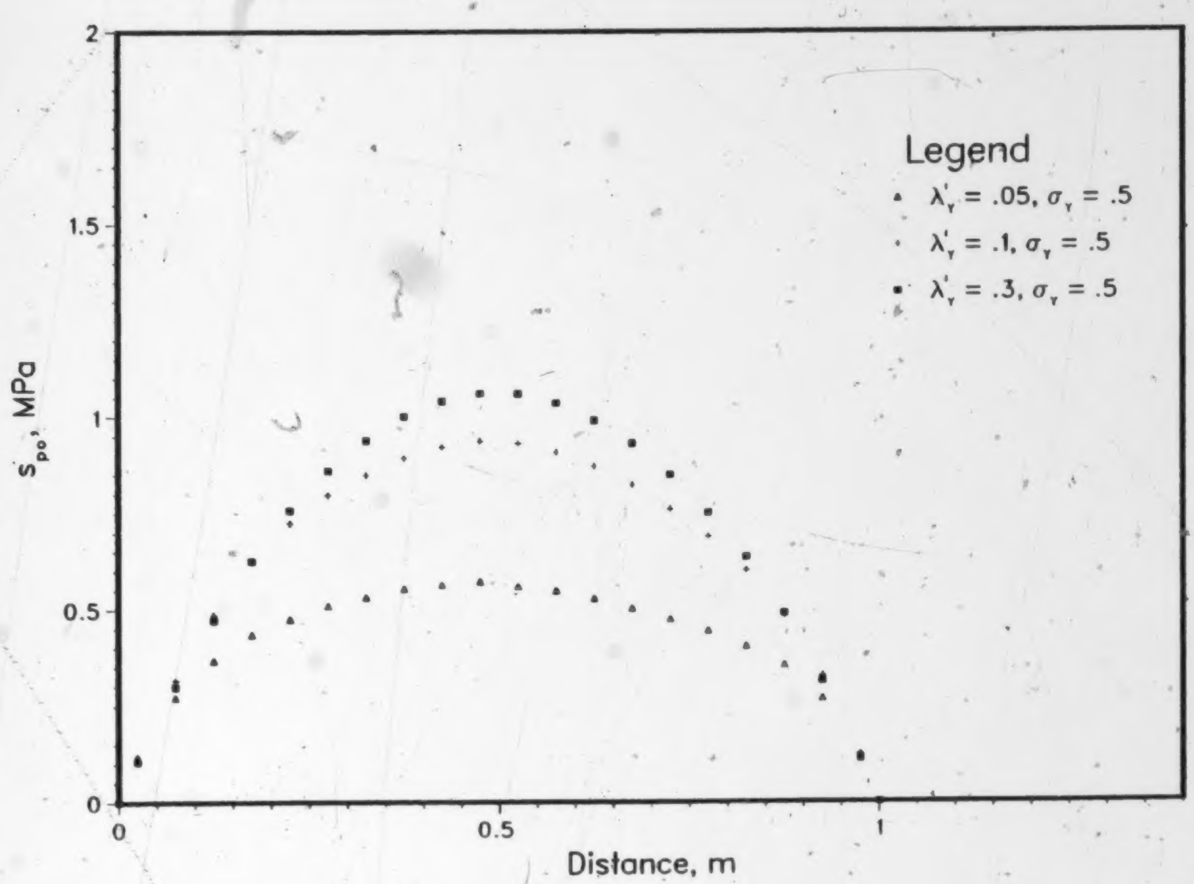


Figure VII-24 Effect of λ'_γ on standard deviation of oil pressure (constant pressure at both ends).

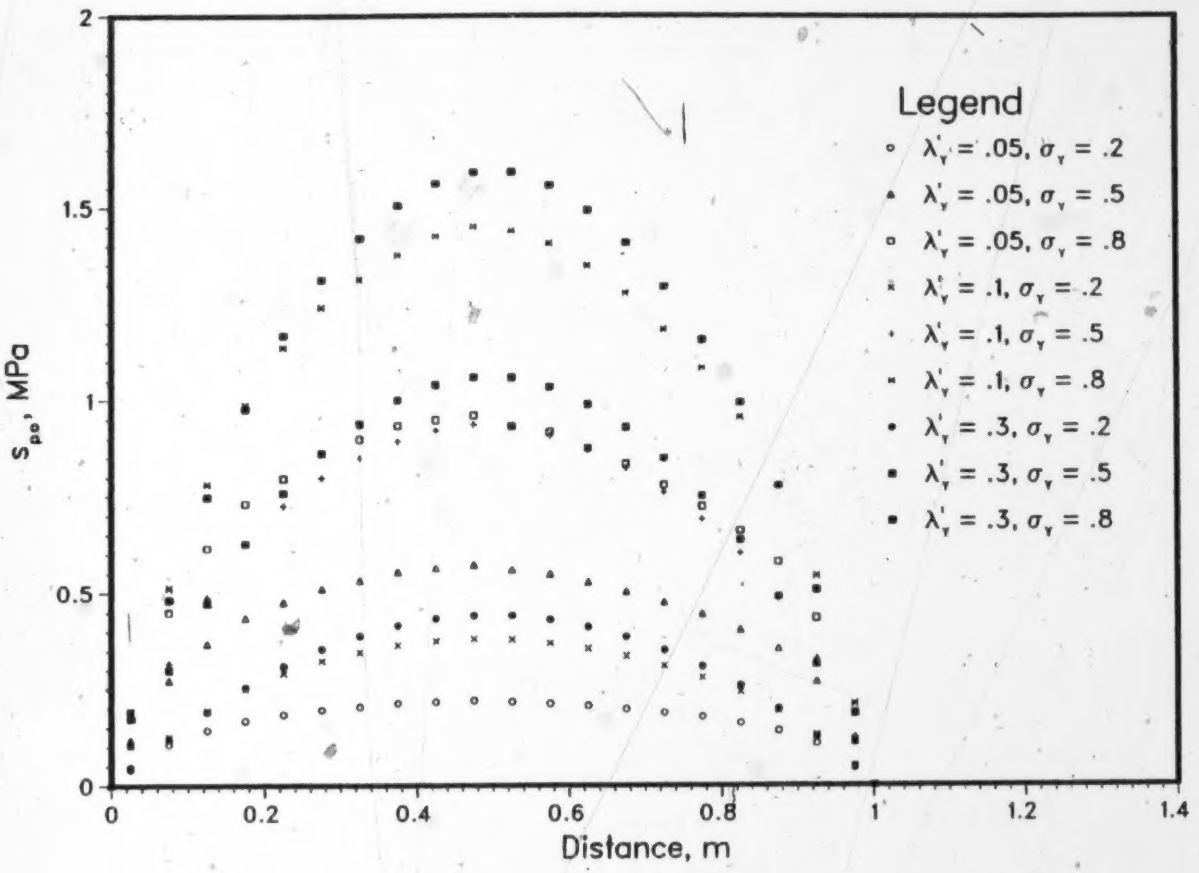


Figure VII-25 Effect of σ_γ and λ'_γ on standard deviation of oil pressure (constant pressure at both ends).

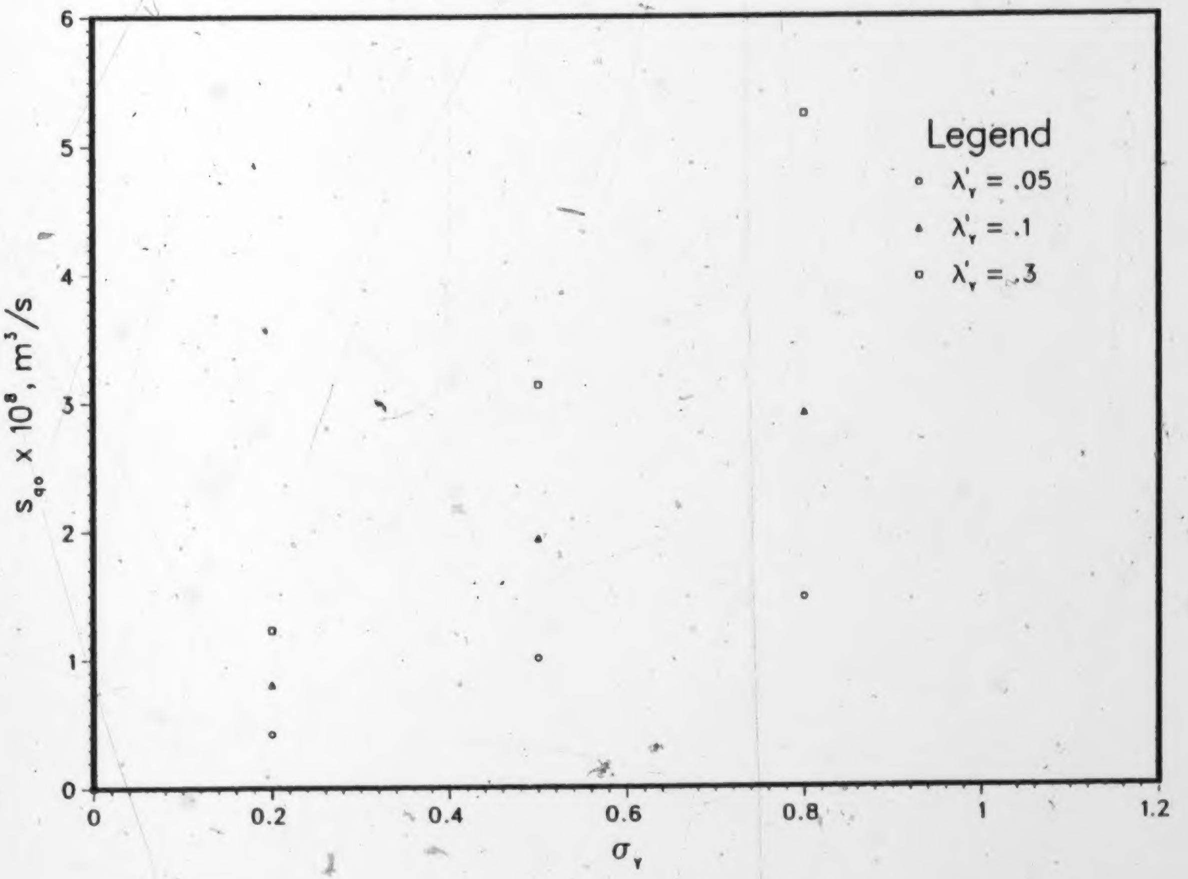


Figure VII-26 Effect of σ_γ and λ'_γ on standard deviation of oil flow rate.

system will be discussed later.

Effect of σ_Y and λ_Y' on the standard deviation of water saturation

The effect of σ_Y and λ_Y' on s_{S_w} for this kind of boundary condition is virtually the same as was found in the constant rate cases. The parameter λ_Y' has no effect on s_{S_w} while σ_Y is responsible for a change in magnitude of s_{S_w} (Figure VII-27). The scattering of data points at high values of σ_Y and λ_Y' is believed to be due to an insufficient number of realizations used at high values of σ_Y and λ_Y'. The value of s_{S_w} is also approximately proportional to σ_Y. Moreover, s_{S_w} is approximately constant along the system. Therefore, it can be said that the type of boundary conditions does not affect s_{S_w}.

Effect of fractional flow

Fractional flow, f_w, does not have an effect on s_{P_o} (Figure VII-28). This is because the pressure gradient is kept constant. It appears that s_{P_o} at each location is proportional to the product $J\sigma_Y\lambda_Y'$, where J is the mean pressure gradient. This result partly supports the analytical expression for s_{P_o} for an unbounded domain, Eq. (IV-49). Figure VII-29 shows the effect of f_w on s_{S_w}. The value of s_{S_w} increases as f_w or $\bar{S}_w$ decreases. The reason for this behavior is the same as given before. (See the section on effect of fractional flow for the constant flow rate

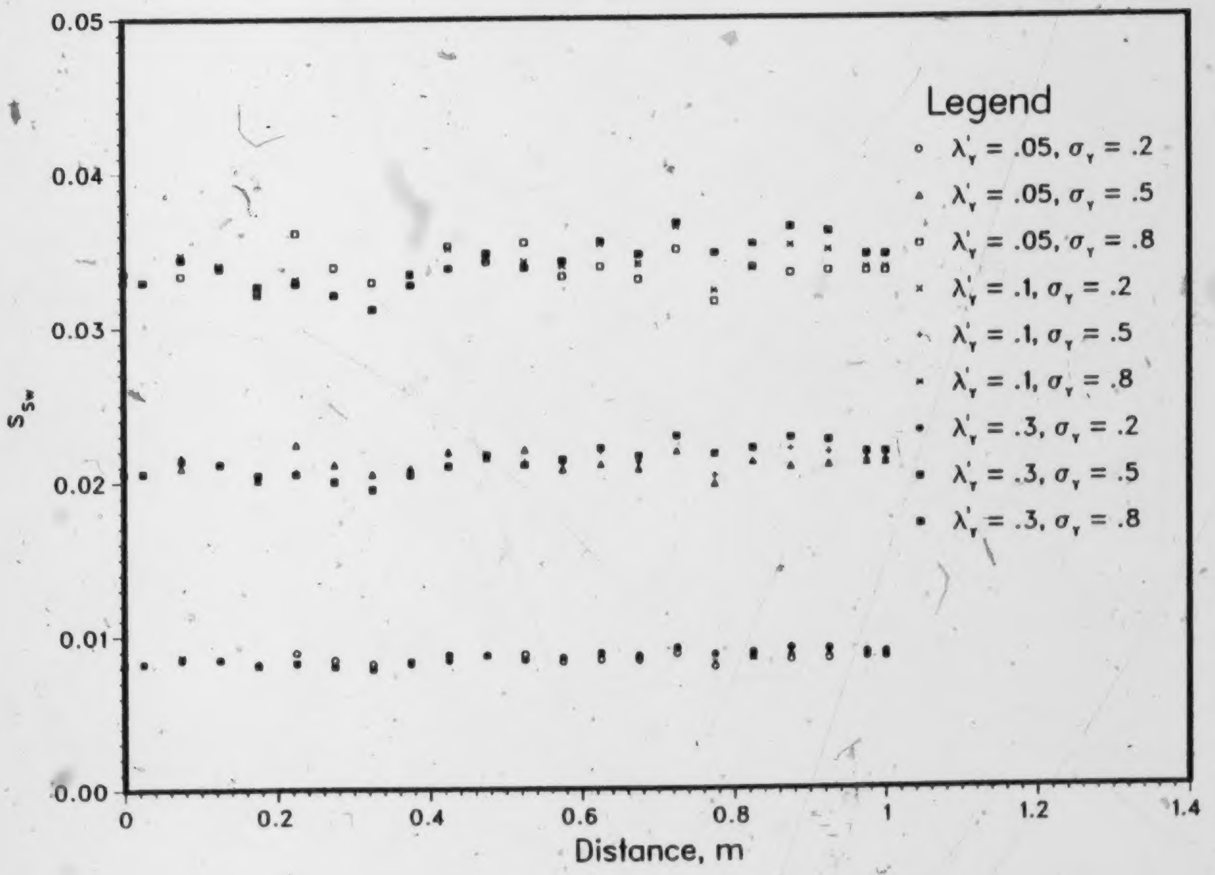


Figure VII-27 Effect of σ_Y and λ'_Y on standard deviation of water saturation (constant pressure at both ends).

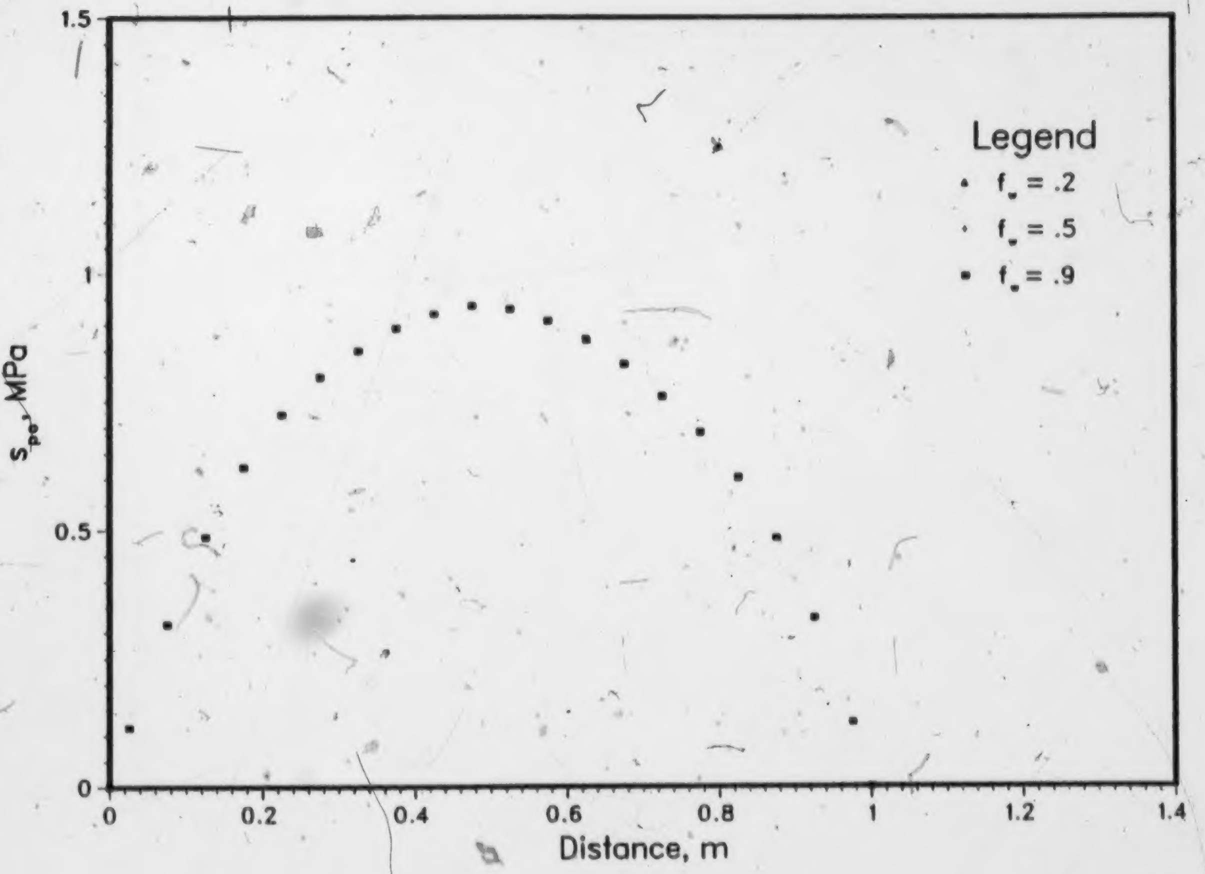


Figure VII-28 Effect of fractional flow on standard deviation of oil pressure (constant pressure at both ends).

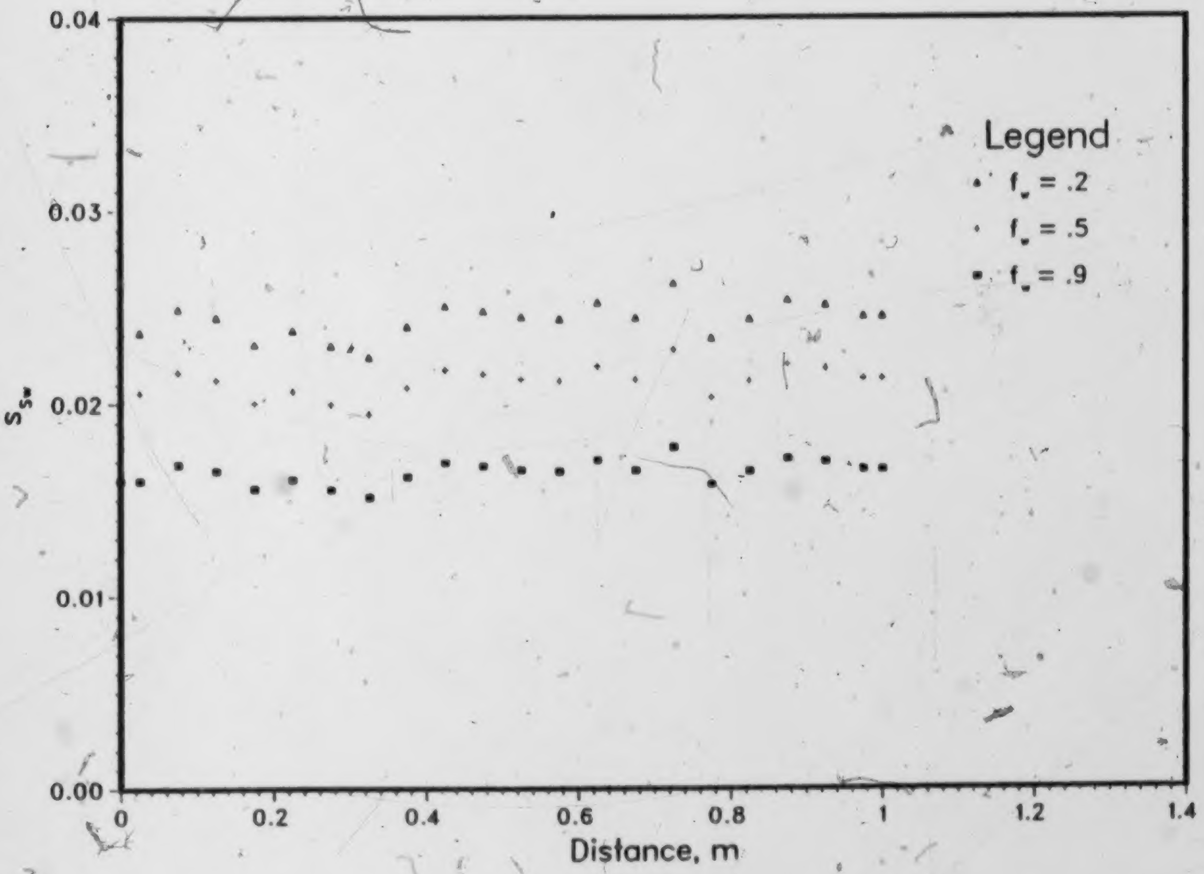


Figure VII-29 Effect of fractional flow on standard deviation of water saturation (constant pressure at both ends).

cases.)

Effect of mobility ratio and capillary pressure

All the results described are obtained by using a viscosity ratio of five, which corresponds to an unfavorable case, and zero capillary pressure. As in the case of constant rate and constant pressure at the outlet end, a change of viscosity ratio to one, which corresponds to a favorable case, and including the effect of capillary pressure do not affect the general conclusions concerning the effects of σ_Y and λ_Y' on $\overline{p}_o$, $\overline{S}_w$, s_{p_o}, and s_{S_w}. This conclusion is based on the results from a few verification runs.

D. Significance of problem formulation

It has been shown that the statistical formulation of the problem can affect the effective permeability estimated. Now it will be shown that a one-dimensional system does not necessarily give a higher value of standard deviation of pressure than that obtained for a two- or three-dimensional system.

A one-dimensional parallel system is used for this purpose. Here it is assumed that the flow in the parallel system is under a constant pressure gradient, i.e. the pressures at both ends are fixed and known with certainty. If this is the case, there will be no uncertainty in pressure at any point in the parallel system, because the pressure in each block (or layer) will be equal for all realizations. On the other hand, if the flow through the parallel system is under a constant rate condition, there will be uncertainty in pressure at points along the flow direction. These arguments lead to the conclusion that a reduction in standard deviation of pressure does not depend only on an increase in the dimensionality of the system, as is well known among investigators, but also on the problem formulation, i.e. how flow rate and pressure gradient are treated. An example has been shown where the flow in a one-dimensional parallel system will yield zero standard deviation of oil pressure under a constant pressure gradient. Therefore, the correct formulation of the problem is essential in order to obtain meaningful results.

E. Implication of the results

Subject to limitations implied by the assumptions made in the Monte Carlo study, the following can be stated. From the results for steady-state flow, it can be concluded that only s_{p_o} depends on λ_Y' or the spatial correlation in rock properties. The parameters $\bar{p}_o$, $\bar{S}_w$, and s_{S_w} are practically independent of λ_Y'. However, all four parameters are dependent on σ_Y or local variability. This gives rise to the useful conclusion that although λ_Y' is an important parameter in determining flow phenomena in porous media, σ_Y is a more important one. A more useful finding for two-phase flow in porous media is that neither the mean nor the standard deviation of the water saturation depend on the spatial correlation. This implies that determination of saturation can be undertaken accurately without a knowledge of the spatial correlation of the porous medium properties.

Another important finding is that reduction in the standard deviation of pressure does not depend only on the dimensionality of the flow system but that it also likely depends on the statistical formulation of the problem. It would be interesting to see whether the dimensionality of the flow system has any influence in reducing the standard deviation of pressure when the correct statistical formulation of the problem is used. Nevertheless, no matter what the result would be, it is still important to recognize that the field boundary conditions should be used in the formulation of the problem, if they are known.

Concerning the concept of effective permeability, it is also important to use the correct statistical formulation of the problem. It has been shown that for one-dimensional serial systems the flow rate should be treated as a deterministic variable while for one-dimensional parallel systems the pressure gradient should be treated as a deterministic variable.

The small variation of water saturation suggests the possibility that (averaged) relative permeability may depend only slightly on heterogeneity of rock properties. This topic will be discussed in the following section.

F. Estimate of relative permeability

It has been shown that for a one-dimensional serial system, the effective permeability to oil can be estimated by using the relative permeability calculated at the mean value of the water saturation, using Eq. (VII-5) which is rewritten here,

$$k_{eff,o} = \bar{k}_{ro}\ k_G \exp(-\tfrac{1}{2}\sigma_Y^2), \qquad \text{(VII-5)}$$

In this equation the effective permeability to oil may be divided into two parts: the relative permeability, $\bar{k}_{ro}$, and the effective (absolute) permeability, $k_G \exp(-\frac{1}{2}\sigma_Y^2)$. If it can be postulated that $\bar{k}_{ro}$ depends on the heterogeneity of rock properties to a much lesser extent than the effective permeability, Eq. (VII-5) can be modified to be applicable to an any dimensional system as

$$k_{eff,o} = \bar{k}_{ro}\ k_{eff,abs}, \qquad \text{(VII-7)}$$

where $k_{eff,abs}$ is the effective (absolute) permeability for the appropriate dimension of the system under consideration.

Now let us consider a heterogeneous reservoir whose σ_Y and λ_Y' are assumed to be known. The relative permeability, k_{ro}, is wanted. Let us assume that many blocks or domains can be cut out from the reservoir and each domain consists of variation of permeability in all three directions. It is further assumed that each domain is not correlated to the others. One indication of a lack of correlation between domains is that the distance between each sampling point (or location) is many times the length of the integral scale. By the assumption of ergodicity (in mean), i.e. the space average is equal to the ensemble average, each domain is equivalent to each realization in a Monte Carlo simulation technique. The flow experiment is performed for all domains to obtain $k_{eff,o}$ at a specific f_w or $\bar{S}_w$. Consequently, $\bar{k}_{ro}$ can be obtained by using these results and Eq. (VII-7) for that specific $\bar{S}_w$. The value of $\bar{k}_{ro}$ at other values of $\bar{S}_w$ can be obtained in a similar manner.

If it can be assumed that the blocks or domains sampled consist of variation of permeability (and/or porosity) in one direction only, such as a long core, a similar procedure can still be used. In this case, Eq. (VII-5) can be used to estimate $\bar{k}_{ro}$. Now the question is how many cores are needed to obtain a good approximation of $\bar{k}_{ro}$. This will be answered next.

The number of cores needed will depend on the degree of heterogeneity of the reservoir. If the degree of heterogeneity is high, the number of cores needed will be high. The estimation of the number of cores needed is related to how accurate an estimate of $\bar{k}_{ro}$ or $k_{eff,o}$ is desired. The value of $k_{eff,o}$ is obtained through the value of the mean oil pressure at the inlet end, which is obtained from the experiment. Let us say that the accuracy of $\bar{k}_{ro}$ is specified in such a way that the mean of the oil pressure at the inlet end is no more than 1% different from the true mean value. Then, the number of cores needed can be calculated from[50,149]

$$n = \left[\frac{z_{1-\alpha/2}\ \sigma'}{E} \right]^2, \qquad \text{(VII-8)}$$

where n is the number of cores needed, E is the maximum allowable error in estimating $E\{\overline{p}_o\}$ (here it is 1% of $E\{\overline{p}_o\}$), $z_{1-\alpha/2}$ is the $100(1-\alpha/2)$ percent point of a standard normal distribution, $1-\alpha$ is the desired probability or confidence level that $\overline{p}_o$ does not differ from $E\{\overline{p}_o\}$ by more than ±E, and σ' is an initial estimate of the process standard deviation, σ_{p_o}. It should be emphasized here that the number of cores, n, calculated by Eq. (VII-8) is only an approximation.

To use Eq. (VII-8), two more things need to be specified, α and σ'. Let us say that the desired confidence level is $1-\alpha = 0.95$ or $\alpha = 0.05$, then $z_{1-\alpha/2} = 1.96$. The parameter σ' can be obtained from Monte Carlo runs like the ones used in this study by specifying all the input parameters to be the same as the coreflood experiment. If it is assumed that the flow condition used in the coreflood experiment is the same as the one used in this study for Monte Carlo techniques and the reservoir has $\sigma_Y = 0.2$, $\lambda_Y = 0.1$ m (assuming that the length of the core is 1 m), the number of core samples needed can be estimated as follows.

For the Monte Carlo results from this study, at $\sigma_Y = 0.2$ and $\lambda'_Y = 0.1$, $s_{p_o} = 1.99340$ MPa and $E\{\bar{p}_o\} = 22.5130$ MPa. It is assumed here that the mean and standard deviation of the oil pressure obtained from the Monte Carlo runs represent the true values. Then, assuming a variation of less than ±1% 95% of the time, E is 0.225130 MPa and using Eq. (VII-8),

$$n = \left[\frac{(1.96)(1.99340\times10^6)}{0.225130\times10^6} \right]^2$$

$$n \approx 301.$$

This is the number of core samples required to obtain the relative permeability to oil that gives the difference of the estimated mean oil pressure from the true mean to be no more than 1%. This value of n is rather high for petroleum engineering practice. It can be made smaller by using a longer core and/or relaxing the requirement for the accuracy of the estimated mean of oil pressure. The former strategy will lower the value of σ' (due to smaller λ'_Y) and the latter will increase the value of E. If the length of the core is 2 m and $\bar{p}_o$ is allowed to be 5% different from $E\{\bar{p}_o\}$,

$$n = \left[\frac{(1.96)(1.04454\times10^6)}{1.121585\times10^6} \right]^2 \approx 3.3 \approx 4$$

This value of n is significantly lower than the previous case. However, the accuracy of the relative permeability curve would also be lower.

The main problem associated with estimating n in a real life situation is that σ' and E (or mean inlet oil pressure) are not known. This problem can be avoided by estimating them. The value of E can be estimated by undertaking a number of experiments with a number of cores to obtain an estimate of the mean of the inlet oil pressure, from which E can be determined. To estimate σ', more cores would be needed. Then, σ' can be estimated from the values of the inlet pressures. With flow experiments on more cores, the estimation of the mean inlet oil pressure will also be better.

The number of cores used in estimating the mean inlet oil pressure (and finally E) and σ' is dependent on the accuracy desired by each individual experimenter. If he undertakes flow experiments on more cores, he would obtain better estimated values for the mean inlet oil pressure and σ'. However, there would be a constraint with respect to expenditure and one would have to use a reasonable number of cores for the flow experiments. In addition, after all, only approximate values of the mean inlet oil pressure and σ' are required so that one can estimate the number of cores needed in the determination of the mean inlet oil pressure at the required accuracy and confidence interval. This is similar to the determination of sample size prior to the actual running of the Monte Carlo simulations. In the Monte Carlo simulations, the information that is needed in order to specify the number of runs (or simulations) is such as to make the simulations unnecessary in the first place.[149] This means that in the Monte Carlo simulations, usually, the unknowns sought are the mean and standard deviation of the output variable (here it is oil pressure) and these two parameters are, in turn, what is needed in the determination of sample size in the Monte Carlo simulations.

However, knowing the number of cores (or sample size) is useful. Because knowing that the number of cores required to accurately estimate the average relative permeability is larger than the number of cores one can afford to undertake flow experiments with will be an indication that the obtained value of average relative permeability to oil should be used with caution.

In the method described, it is implicitly assumed that the variation in saturation is (relatively) smaller than the variation in pressure. Therefore, the accuracy in estimating the saturation will always be higher than the accuracy of estimation of pressure, given the same number of cores.

If the values of σ_Y and λ'_Y are known, Tables VII-11 and VII-12 can be used to estimate the number of cores needed to estimate the average relative permeability to oil. Table VII-11 is based on the assumption that the estimated mean oil pressure is no more than 1% different than the true mean, while Table VII-12 is based on the assumption that the estimated mean oil pressure is no more than 5% different than the true mean. The confidence level for both tables is

Table VII-11

Number of cores needed for 1% allowable error in estimating mean pressure

Case	λ'_Y	σ_Y	σ' (MPa)	E (MPa)	n
1	homogeneous		-	-	1
2	0.05	0.20	1.04454	0.224317	84
3	0.05	0.50	3.02831	0.249955	564
4	0.05	0.80	6.49316	0.304532	1747
5	0.10	0.20	1.99340	0.225130	302
6	0.10	0.50	5.65953	0.252208	1935
7	0.10	0.80	11.74880	0.308880	5558
8	0.30	0.20	3.00930	0.226265	680
9	0.30	0.50	8.58029	0.255574	4330
10	0.30	0.80	17.63970	0.315452	12013

Table VII-12

Number of cores needed for 5% allowable error in estimating mean pressure

Case	λ'_Y	σ_Y	σ' (MPa)	E (MPa)	n
1	homogeneous		-	-	1
2	0.05	0.20	1.04454	1.121585	4
3	0.05	0.50	3.02831	1.249775	23
4	0.05	0.80	6.49316	1.522660	70
5	0.10	0.20	1.99340	1.125650	12
6	0.10	0.50	5.65953	1.277540	76
7	0.10	0.80	11.74880	1.544400	223
8	0.30	0.20	3.00930	1.131325	28
9	0.30	0.50	8.58029	1.277870	174
10	0.30	0.80	17.63970	1.577260	481

95%. It should be emphasized that Tables VII-11 and VII-12 were constructed based on the assumption that $\overline{k}_{ro}$ depends on the heterogeneity of rock properties to a much lesser extent than does the effective absolute permeability. In addition, the tables are constructed using the values of $\overline{p}_o$ and s_{p_o} obtained from the constant rate cases with zero capillary pressure and the unfavourable mobility-ratio condition.

The number of cores needed in Table VII-11 is very large and it is impractical to do flow experiments for such a large number of cores. This table is shown to illustrate the effect of heterogeneity on the number of cores needed to estimate the relative permeability to oil with a 1% allowable error in estimating mean pressure. Table VII-12 shows that if one is willing to accept less accuracy in the value of $\overline{k}_{ro}$, the number of cores needed can be reduced considerably. Table VII-11 (or VII-12) should not be used to estimate the number of cores needed, if the assumptions underlying the estimation of $\overline{p}_o$ and s_{p_o} are markedly different from those used in this study.

In the above application, each core is interpreted to be a realization. This may be called a small scale application. Another possible application is a large scale application. Here a realization may be interpreted as a portion of a reservoir between two wells. If the values of σ_Y, λ_Y', and fluid and rock-fluid interaction properties are known, the Monte Carlo simulation runs described earlier can be undertaken using the real boundary conditions which pertain in the field. (Note that the permeability and porosity at individual locations are not known.) The means and standard deviations of pressure at the wells can be obtained. These mean and standard deviation values can be used to obtain a rough estimate of the pressure that can be expected when the reservoir is operated. Another application is that a portion of a reservoir may be modelled and Monte Carlo techniques used to obtain values of the mean and standard deviation of pressure. These values of standard deviation of pressure will give a basis which is useful, later on, for history matching using a deterministic model.

In all the applications described above, it is implicitly assumed that log-permeability and porosity are statistically homogeneous and statistically isotropic. It is also assumed that the variation in rock properties is uniformly spread in three-dimensional space. That is, it is not a stratified or composite reservoir.

G. Test of independence and normality

Values of output random variables from each realization should be independent of one another. This is required so that meaningful statistical analyses of the output random variable can be performed. The independence of values of the output random variables is ensured by using independent random numbers for generation of input random variables.[150] Independence of the random numbers was achieved by using a well-accepted random number generator. In addition, the statistics of the generated values of input random variable were tested against their analytical values. All the test results were satifactory.

Another way of testing the independence of values of the output random variables is to test the correlation of these values. In fact, this test will provide information about the correlation of values of the output random variables, not their independence. If the output random variables are normally distributed, a lack of correlation implies independence of the values of the output random variables. The values of p_o at block 1 (inlet oil pressure) were used for the correlation test for steady-state and transient flows with constant flow rate. For steady-state flows with constant pressure at both ends, the values of p_o at the middle block (block 12) were used for this purpose. The correlation coefficients (the ratio of covariance and variance; note that all p_o have the same variance) of p_{oi} and p_{oi+1} up to p_{oi} and p_{oi+10}, where i is the realization number, are less than 0.1 for all the cases considered. In fact, most of the correlation coefficient values are lower than 0.05. These low values of the correlation coefficient imply that the p_o values of each realization are almost uncorrelated with one another.

The same set of values of p_o were used to test for normality of the output random variables. The graphical method called the stabilized probability plot[151] was used. From the results of this plot, the values of p_o for most cases have a normal distribution at the 95% confidence level (or at the 0.05 level of significance). Figure VII-30 is an example of the stabilized probability plots. The middle line is the line where the data values are all equal to the theoretical values of a normal random variable. The two boundary-lines show the 95% confidence interval. If all the plotted points fall within these two boundary-lines, the data values can be interpreted as having a normal distribution at the 95% confidence level. The values for the abscissa and the ordinate are the transformed values of the theoretical and data values, respectively. This transformation is undertaken so that the variances of the plotted points are approximately equal. This results in straight boundary-lines for the confidence interval.

The only two plots that indicate non-normality of the values of p_o are for the cases with $\lambda'_Y = 0.3$ and $\sigma_Y = 0.8$ for both steady-state and transient flows with constant flow rate. Figure VII-31 shows the plot for oil pressure of the steady-state flow with constant flow rate for $\lambda'_Y = 0.3$ and $\sigma_Y = 0.8$. Only a few plotted points on the lower left-hand side of the graph deviated beyond the 95% confidence interval. The plot for transient flow with constant flow rate for $\lambda'_Y = 0.3$ and $\sigma_Y = 0.8$ is similar to the one shown in Figure VII-31. All the plots for cases with constant pressure at both ends show normality of p_o.

Because p_o's for most cases are normally distributed and independent, the statistical analyses for independent

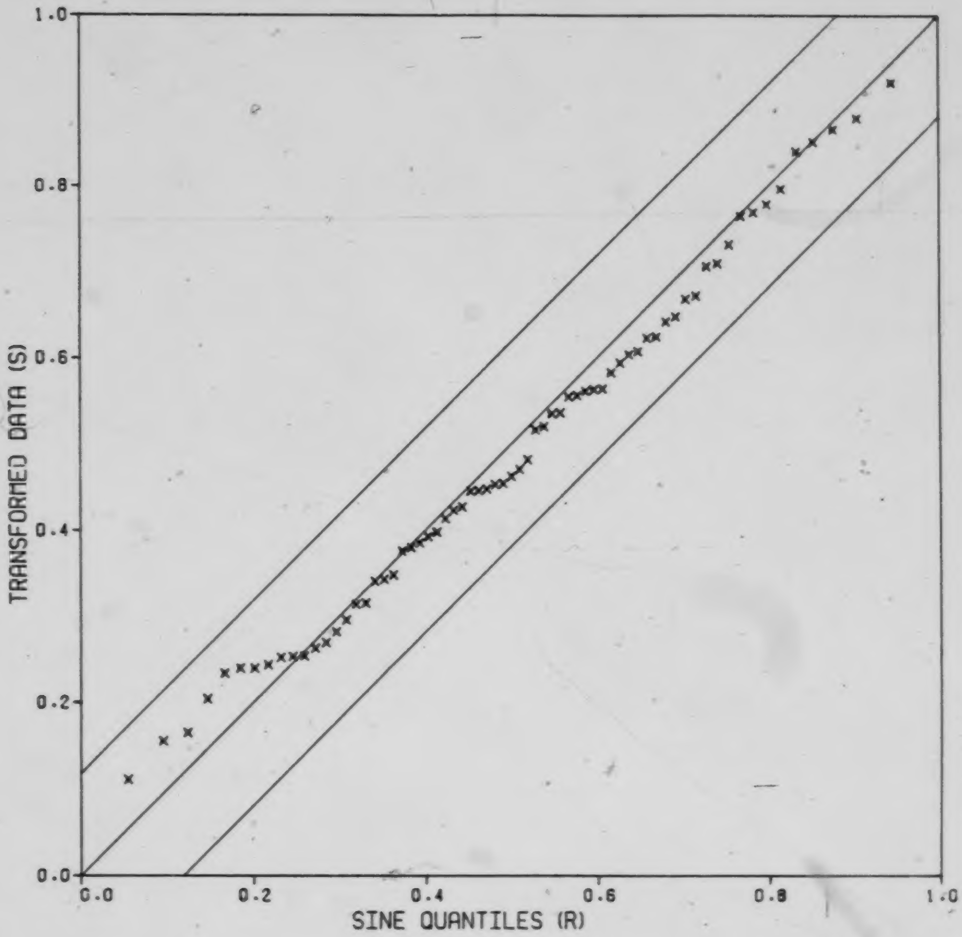


Figure VII-30 The stabilized probability plot for oil pressure of steady-state flow with constant rate, $\lambda_Y'=0.3$, $\sigma_Y=0.2$.

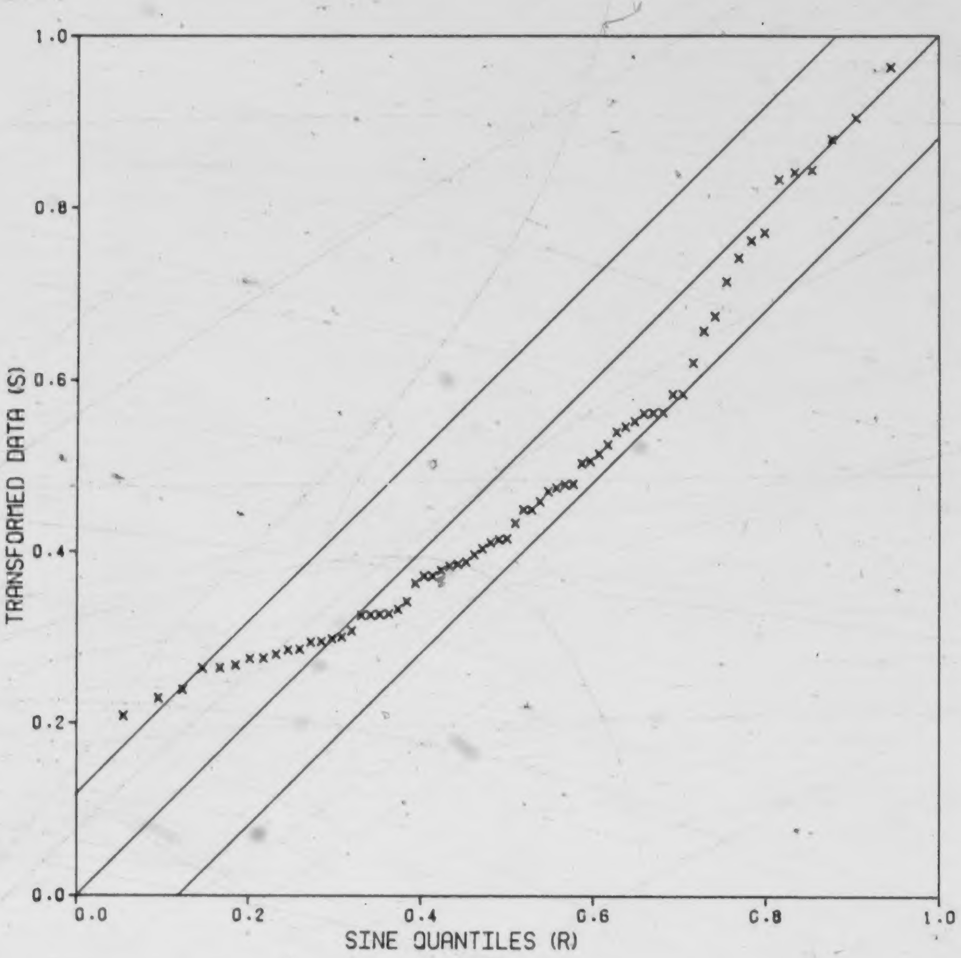


Figure VII-31 The stabilized probability plot for oil pressure of steady-state flow with constant rate, $\lambda_Y' = 0.3$, $\sigma_Y = 0.8$.

normally distributed variates can be used. The only exception is for the cases of the highest degree of heterogeneity considered, i.e. $\lambda'_Y = 0.3$ and $\sigma_Y = 0.8$.

H. Regression analysis of oil pressure difference

In order to confirm the conclusion that $\overline{p}_o$ depends on σ_Y only and not on λ'_Y, a regression analysis of oil pressure difference (between the inlet and outlet oil pressures) was undertaken. The stepwise regression[152-154] procedure was used. The computation was undertaken using the BMDP statistical software.[155] The independent variables used were σ_Y, σ_Y^2, λ'_Y, and λ'^2_Y while the dependent variable was the oil pressure difference. This selection was based on Darcy's equation and the analytical expression for the effective permeability. In addition, prior to the application of regression analysis, the oil pressure difference was transformed to the logarithm of the oil pressure difference. This transformation was based also on the analytical expression for the effective permeability. Consequently, the following model was used for the regression analysis,

$$\ln(\Delta p_o) = b_0 + b_1\sigma_Y + b_2\lambda'_Y + b_3\sigma_Y^2 + b_4\lambda'^2_Y, \qquad \text{(VII-9)}$$

where Δp_o is the oil pressure difference.

It is evident from Figure VII-11 that the standard deviation (or variance) of oil pressure changes with changes in the values of σ_Y and λ'_Y. This indicates that weighted least squares regression is more appropriate for analyzing the relationship between oil pressure difference and σ_Y and λ'_Y.[153,155] The weighting function will result in equality of variance of $\ln(\Delta p_o)$ as required by regression. Because the variance of oil pressure is approximately proportional to $\sigma_Y^2\lambda_Y^2$ (see Eq. (VII-6)), a weighting function of the form $1/(\sigma_Y^2\lambda_Y'^2)$ was used. Data from the constant rate, steady-state flow cases were used in this regression analysis. The weighted stepwise regression was undertaken with the model in Eq. (VII-9). The only significant term retained in the regression was σ_Y^2. The resultant prediction equation was

$$\ln(\Delta p_o) = b_0 + b_3\sigma_Y^2. \tag{VII-10}$$

The standard error (square root of variance) of estimate for $\ln(\Delta p_o)$ was 0.098. The intercept estimate, b_0, was 16.90. The mean estimate for b_3 was 0.465 with a standard error of 0.011.

From Eq. (VII-10) it can be seen that only σ_Y and not λ'_Y has influence on the predicted value of $\ln(\Delta p_o)$. Because the outlet oil pressure was kept constant for the constant rate, steady-state flow cases, it can be concluded that the predicted inlet oil pressure depends upon σ_Y only and not upon λ'_Y. This conclusion supports the previous conclusion that the inlet $\overline{p}_o$ depends upon σ_Y only.

In fact, using Darcy's equation and the analytical expression for the effective permeability, it can be shown analytically that the expected value of $\ln(\Delta p_o)$ depends on σ_Y only. Let us start with an equation of the form,

$$\Delta p_o = -\frac{q_o \mu_o \Delta L \epsilon}{k_{ro} k_{eff} A}, \qquad \text{(VII-11)}$$

where ϵ is a multiplicative random error (see Ref. 153), and other terms have been defined before. Eq. (VII-11) is a modified Darcy's equation which includes the multiplicative random error, ϵ. Now using the expression of k_{eff} for a serial system (see Appendix A), Eq. (VII-11) becomes

$$\Delta p_o = -\frac{q_o \mu_o \Delta L \epsilon}{k_{ro} A k_G \exp(-\sigma_Y^2/2)}. \qquad \text{(VII-12)}$$

Taking the logarithm of Eq. (VII-12),

$$\ln(\Delta p_o) = C_1 + \frac{1}{2}\sigma_Y^2 + \ln(\epsilon), \qquad \text{(VII-13)}$$

where $C_1 = \ln(-q_o \mu_o \Delta L / k_{ro} A k_G)$. Then, taking the expectation, Eq. (VII-13) becomes

$$E\{\ln(\Delta p_o)\} = C_1 + \frac{1}{2}\sigma_Y^2, \qquad \text{(VII-14)}$$

where it has been assumed that the input variance of $\ln(k)$, σ_Y^2, is known without uncertainty and $E\{\ln(\epsilon)\} = 0$. It has been assumed also that k_{ro} is not a random variable or, in other words, the dependence of k_{ro} on heterogeneity in rock properties is insignificant.

Using input data, C_1 was calculated to be 16.90. Comparing Eqs. (VII-10) and (VII-14), it can be seen that the magnitudes of the constant term in these two equations are the same (within four significant digits). The difference between coefficients of σ_Y^2 of these two equations is evident. However, with the standard error of the coefficient of σ_Y^2, which is equal to 0.011, it can be said that the true value of the coeficient of σ_Y^2 lies in the interval (0.487 to 0.443) with 95% confidence. The possible upper-bound value (0.487) of the true value of the coefficient of σ_Y^2 is very close to 0.5, the theoretical coefficient value. One explanation that the interval of the true value of the coefficient of σ_Y^2 obtained from the regression analysis does not cover the theoretical value, which is equal to 0.5, is that the number of Monte Carlo runs for the cases of high heterogeneity was not sufficient. This caused the inlet oil-pressure values obtained from the Monte Carlo simulation to fail to cover all of the possible values of the inlet oil-pressure (evenly) for cases with a high degree of heterogeneity. This led to the discrepancy between the regression equation, Eq. (VII-10), and the theoretical equation, Eq. (VII-14). However, from a practical point of view, this discrepancy was small. Therefore, both the regression equation and the theoretical equation confirm that the mean of inlet oil pressure is dependent on the standard deviation of ln(k) only and not on the integral scale of ln(k).

It should be noted that in the regression analysis the input values of σ_Y and λ'_Y were used. However, it has been shown that the estimates of variance and integral scale of generated ln(k) values are biased (see Eq. (V-7) and Eqs. (V-11) to (V-14)) due to correlation between ln(k) values. It has been suggested that by using the estimates of variance and integral scale of generated ln(k) values instead of the input values of variance and integral scale in the model used for regression analysis, the resultant prediction equation might be more appropriate.[156]

VIII. TRANSIENT FLOW

Transient flow is studied for the cases with constant flow rate and constant pressure at the outlet end only. These boundary conditions are identical to the ones used in the external-drive techniques. The necessary data are presented in Table VIII-1. The flow rates used are such that the instability number is less than π^2 to ensure stability.[124] The realizations of permeability and porosity values are the same as those used in the previous chapter. All results shown are obtained at a time approximately equal to one half of the time to breakthrough.

A. Effect of σ_Y and λ_Y' on the mean of oil pressure

The effect of σ_Y and λ_Y' on $\overline{p}_o$ shows a similar trend to that in the steady-state cases (Figure VIII-1). The parameter $\overline{p}_o$ depends only on σ_Y. It seems that in the frontal region, $\overline{p}_o$ is also affected by λ_Y'. This effect may not be direct but may act indirectly through variations in S_w. In the region ahead of the front the pressure gradient is constant because the water saturation is constant at S_{wirr}. Behind the front, the pressure gradient varies because of a variation in S_w along the system behind the front. In the frontal region, the pressure gradient varies in accordance with the variation in saturation.

Table VIII-1

Data for transient flow

Length of core (m)	1.0
Cross-section area (m^2)	0.0025
Displacement rate (m^3/s)	4×10^{-7}
Outlet pressure (MPa)	0.1
Input μ_Y	-29.94677
Input k_G (m^2)	9.869224×10^{-14}
Input μ_ϕ	0.201
Number of blocks	20

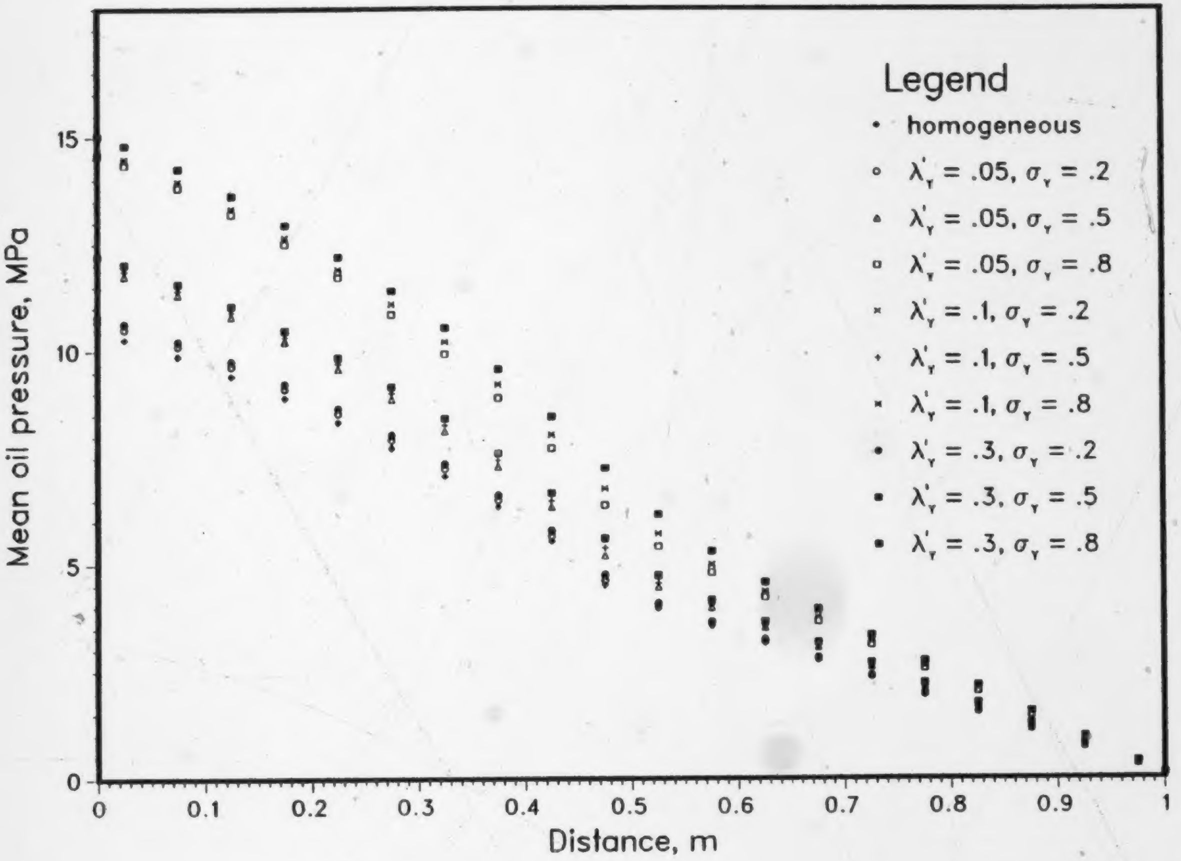


Figure VIII-1 Effect of σ_Y and λ'_Y on mean of oil pressure (transient flow).

B. Effect of σ_Y and λ'_Y on the standard deviation of oil pressure

The general trend of the effect of σ_Y and λ'_Y on s_{p_o} is similar to that for the steady-state cases (Figure VIII-2). The parameter σ_Y has a larger relative effect on s_{p_o} than does λ'_Y. The rate of increase of the value of s_{p_o} increases uniformly as σ_Y increases, while the rate of increase of the value of s_{p_o} decreases as λ'_Y increases. This again shows the importance of local variation over the spatial correlation of rock properties on uncertainty in oil pressure.

C. Effect of σ_Y and λ'_Y on the mean of water saturation

Figure VIII-3 shows the effect of σ_Y on $\bar{S}_w$. As σ_Y increases, the smearing of the front increases. The effect of λ'_Y on $\bar{S}_w$ follows a similar trend (Figure VIII-4). The mean saturation front is more dispersed as λ'_Y increases. In the frontal region, both the effect of σ_Y and that of λ'_Y are significant. This implies that, as the degree of heterogeneity of rock properties increases, the mean saturation is more dispersed. This is consistent with the results of Warren and Skiba[75] that the degree of dispersion increases as the degree of heterogeneity increases.

D. Effect of σ_Y and λ'_Y on the standard deviation of water saturation

The effect of σ_Y and λ'_Y on s_{S_w} for transient flow is shown in Figure VIII-5. From this figure the following

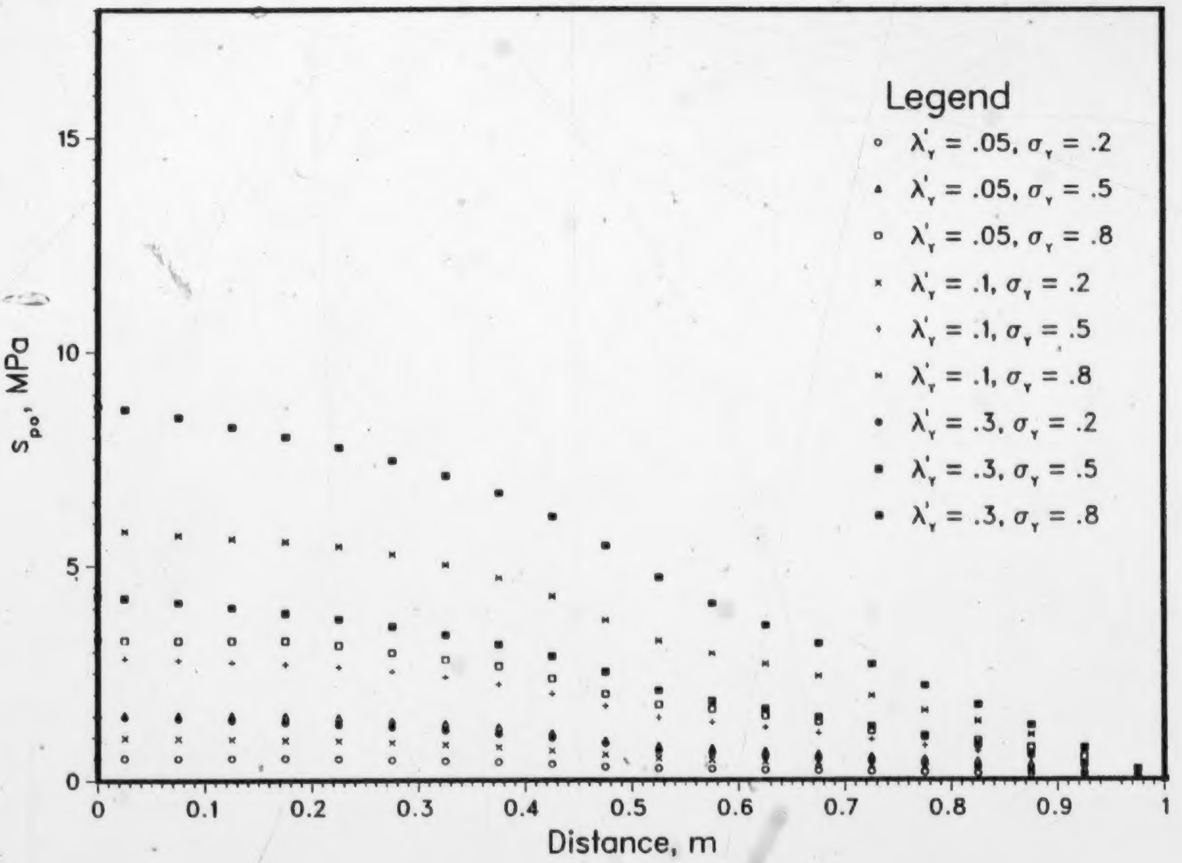


Figure VIII-2 Effect of σ_Y and λ'_Y on standard deviation of oil pressure (transient flow).

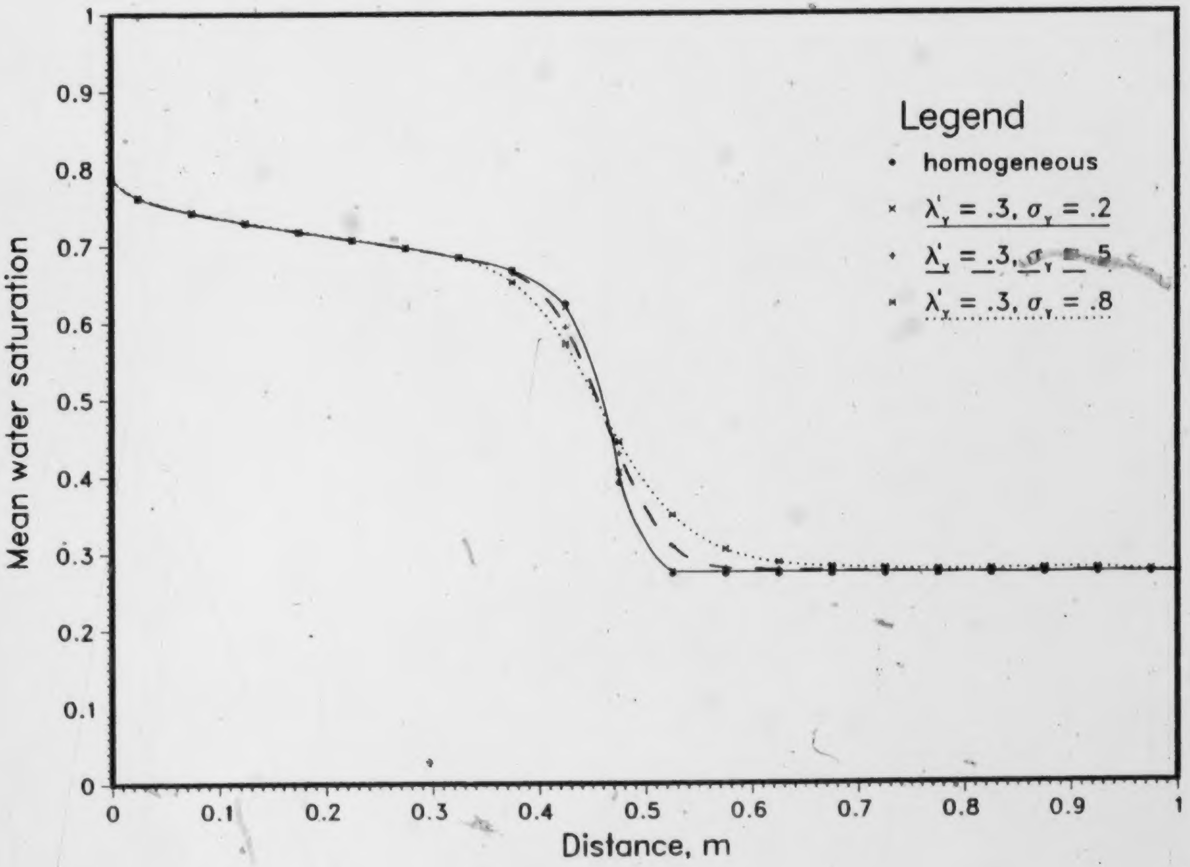


Figure VIII-3 Effect of σ_Y on mean of water saturation (transient flow).

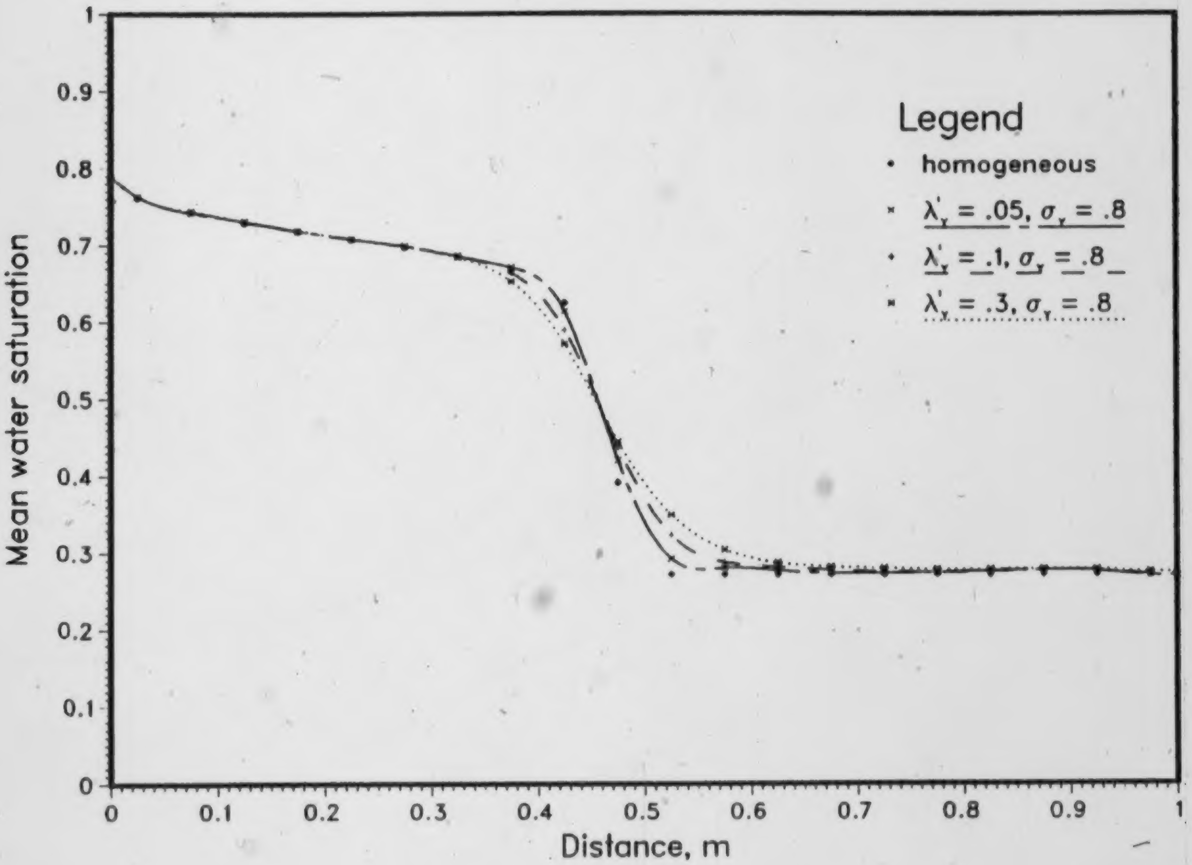


Figure VIII-4 Effect of λ'_Y on mean of water saturation (transient flow).

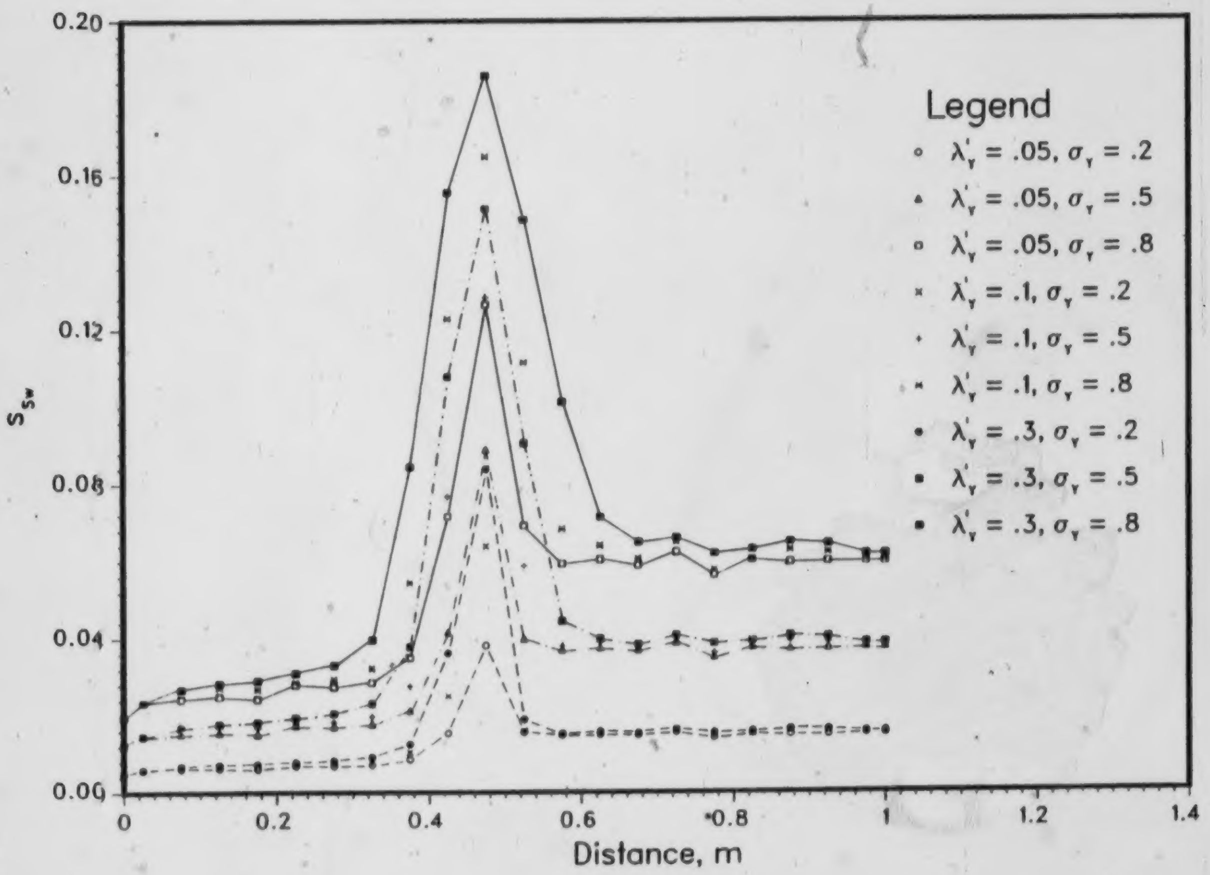


Figure VIII-5 Effect of σ_Y and λ'_Y on standard deviation of water saturation (transient flow).

conclusions can be drawn:

1. The value of s_{S_w} in the regions ahead and behind the front are approximately proportional to σ_Y. The effect of λ_Y' on s_{S_w} is evident in the regions close to the front where s_{S_w} increases as λ_Y' increases.
2. The value of s_{S_w} in the region ahead of the front is higher than that behind the front. This is because in the region ahead of the front $\bar{S}_w$ is lower than behind the front; therefore, the effect of rock heterogeneity is more pronounced in the region ahead of the front.
3. In the frontal region, s_{S_w} is much higher than in the other two regions. The effect of λ_Y' on s_{S_w} is also more pronounced in this region.
4. As the degree of heterogeneity increases, uncertainty in the location of the front increases.

Figures VIII-6 and VIII-7 show the mean saturation profile and the one standard deviation interval for cases of a lower degree of heterogeneity ($\sigma_Y = 0.2$, $\lambda_Y' = 0.05$) and a higher degree of heterogeneity ($\sigma_Y = 0.8$, $\lambda_Y' = 0.3$), respectively. It can be seen clearly that heterogeneity in rock properties affects both the shape and variation of the location of the front.

E. Effect of mobility ratio and capillary pressure

The results decribed above are for the cases with unfavorable mobility ratio ($\mu_o/\mu_w = 5.0$, Mr = 1.6) and zero

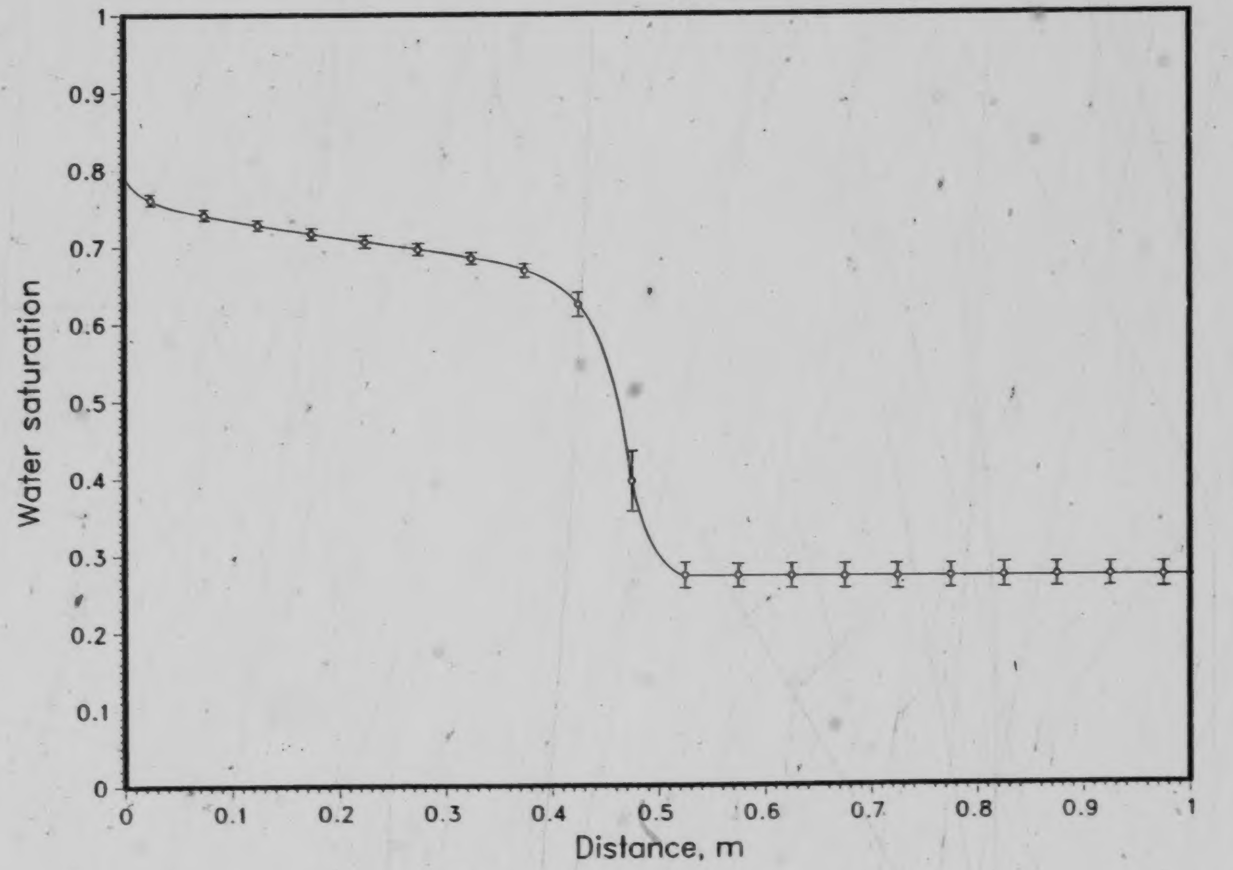


Figure VIII-6 Mean saturation profile and one standard deviation interval, $\sigma_Y = 0.2$, $\lambda'_Y = 0.05$.

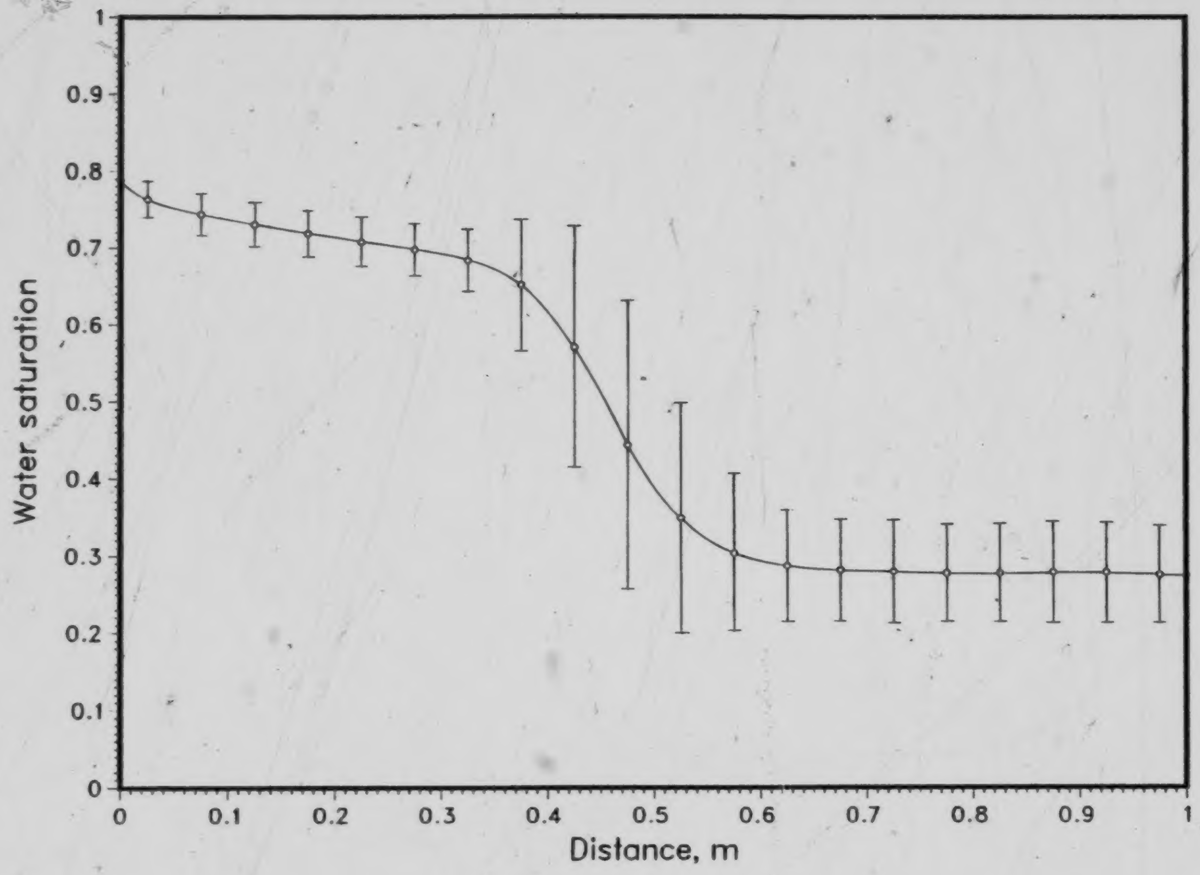


Figure VIII-7 Mean saturation profile and one standard deviation interval, $\sigma_Y = 0.8$, $\lambda_Y' = 0.3$.

capillary pressure. Cases with a favorable mobility ratio μ_o/μ_w = 1.0, Mr = 0.3) and non-zero capillary pressure were also run. Similar conclusions about the effect of heterogeneity on the resulting pressure and saturation as in the cases of unfavorable mobility ratio and zero capillary pressure can be drawn from the results of cases with favorable mobility ratio and non-zero capillary pressure. Results for cases with a favorable mobility ratio and non-zero capillary pressure are not shown.

In general the effect of heterogeneity in rock properties on the resulting pressure and saturation for the transient flow cases is similar to that of the steady-state flow cases. The exceptions are that in the transient flow cases, the effect of λ_Y' on $\overline{p}_o$, $\overline{S}_w$, and s_{S_w} in the frontal region is more pronounced than the effect of λ_Y' on $\overline{p}_o$, $\overline{S}_w$, and s_{S_w} in the steady-state cases.

F. Implication of the results

General conclusions about the effect of heterogeneity in rock properties on the resulting pressure and saturation of the steady-state flow case can be applied here. (These conclusions are subject to limitations implied by the assumptions made in the steady-state cases.) The exception is at the frontal region. In this region, both the local variation (σ_Y) and the spatial correlation (λ_Y') have considerable influence on the mean and standard deviation of the water saturation. This leads to the conclusion that:

1. for small scale applications, many core samples will be required to get an accurate estimate of the average relative permeability for highly heterogeneous reservoirs, when using external-drive techniques; 2. for large scale applications, in highly heterogeneous reservoirs, the uncertainty in location of the flood-front region will be high. This uncertainty must be taken into account if the breakthrough time is important to the field operation (i.e. scheduling for installation of water disposal facilities, etc.); 3. for history matching of waterflood processes, in highly heterogeneous reservoirs, matching any variable in the frontal region may be meaningless, and the matching effort should be directed to the region away from the frontal region; in addition, matching of breakthrough time should be performed with care.

IX. CONCLUSIONS AND RECOMMENDATIONS

The problems of two-phase flow through porous media have been solved by two stochastic methods: spectral and Monte Carlo techniques. The exponential form of the autocorrelation function for log-permeability was used in both techniques. It has been shown that the exponential autocorrelation function for log-permeability is appropriate for flow in porous media problems in one-, two- and three-dimensional systems.

Using spectral techniques and the exponential form of the autocorrelation function for log-permeability, the variance of oil pressure was obtained for a three-dimensional system. The variances of oil pressure for one- and two-dimensional systems and the covariances of oil pressure for one-, two- and three-dimensional systems could not be obtained because of a singularity problem. The singularity problem may have arisen because of the assumption that the perturbation part of the oil pressure is statistically homogeneous and statistically isotropic. In two-phase flow problems the variance of oil pressure depends on the variance and integral scale of log-permeability and the mean oil pressure gradient in the main flow direction. The mean oil pressure gradient is, in turn, dependent on saturation. Major assumptions used in the spectral techniques were: the perturbation terms of log-permeability and oil pressure are small and statistically homogeneous and statistically isotropic. In addition, the flow rate and water saturation can be treated deterministically and the mean of log-permeability is constant.

In solving flow through porous media problems by Monte Carlo techniques, values of permeability and porosity have to be generated. It has been shown that sample variance, sample covariance, and sample integral scale evaluated from the generated values of the variate which are correlated are biased. The expression for the expected value of the sample covariance has been derived. This expression can be used to justify the validity of the generated values of the variate. It has been emphasized that the effect of the correlation of the generated values of the variate is important. When reporting the effect of heterogeneity of rock properties on the resulting pressure and saturation the input values of variance (or standard deviation) and integral scale of log-permeability or porosity should be used. The procedure for generation of values of permeability and porosity has been presented. The following conclusions from Monte Carlo study for both steady-state and transient two-phase flows in bounded domains are subject to assumptions made and to systems used in this study. It should be noted that in all cases the flow was assumed to be stable.

The conclusions from the Monte Carlo study for the steady-state two-phase flow in a bounded domain are as follows.

1. The mean oil pressure depends only on the standard

deviation of log-permeability.

2. The standard deviation of oil pressure depends on both the standard deviation and integral scale of log-permeability. However, the influence of the standard deviation of log-permeability on the standard deviation of oil pressure is more pronounced.
3. The standard deviation and integral scale of log-permeability have an insignificant effect on the mean water saturation.
4. The standard deviation of the water saturation depends only on the standard deviation and not on the integral scale of log-permeability.
5. For the case with constant pressure at both ends the mean and standard deviation of the oil flow rate depend on both the standard deviation and the integral scale of log-permeability.
6. The standard deviation and integral scale of log-permeability have an insignificant effect on the plot of fractional flow versus mean water saturation.
7. Mobility ratio and capillary pressure have an insignificant influence on the effect of the standard deviation and integral scale of log-permeability on the mean and standard deviation of oil pressure and water saturation.
8. The mean and standard deviation of water saturation depend on the standard deviation of log-permeability to a much lesser extent than the mean and standard

deviation of the oil pressure, depending on the standard deviation of log-permeability. Consequently, it is possible to estimate the average relative permeability to oil (or to water) for a heterogeneous system.

9. A reduction in standard deviation of pressure depends not only on an increase in the dimensionality of the system but also on the problem formulation, i.e. how flow rate and pressure gradient are treated.
10. Concerning the concept of effective permeability, it is important to use the correct statistical formulation of the problem.
11. A procedure to estimate the number of cores needed to estimate, at given confidence level, the (average) relative permeability has been proposed.

The conclusions from the Monte Carlo study for transient two-phase flow in a bounded domain are as follows.

1. The effect of standard deviation and integral scale of log-permeability on the mean and standard deviation of oil pressure and water saturation is similar to that of the steady state flow except in the frontal region.
2. In the frontal region, an increase in standard deviation or integral scale (or both) of log-permeability causes smearing of the front to increase.
3. In the frontal region, the standard deviation of water saturation is much higher than in the regions ahead of and behind the frontal region. In addition, the effect of standard deviation and integral scale of

log-permeability on the standard deviation of water saturation is more pronounced in the frontal region.

4. Mobility ratio and capillary pressure have an insignificant influence on the effect of heterogeneity in rock properties on the mean and standard deviation of oil pressure and water saturation.
5. To obtain the average effective permeability to each phase for a highly heterogeneous reservoir, fewer cores would be needed if steady-state coreflood tests were used, instead of external drive techniques.
6. For history matching of waterflood processes, in highly heterogeneous reservoirs, the matching effort should be directed to the region away from the frontal region.

From the results of this study many new, unanswered questions have been raised. This leads to the need for further investigation on this subject. The following investigations are recommended:

1. One of the speculative remarks of this study is that if a sufficient number of realizations is used, the mean water saturation should not depend on the heterogeneity of the rock properties. This remark needs further investigation using a large number of realizations.
2. It has been pointed out that the statistical formulation of the problem is crucial to the proper determination of the effective permeability. It has also been speculated that for two- and three-dimensional systems, it might be necessary to treat both flow rate and pressure gradient

as random variables. It is also suggested that the reduction in the standard deviation of pressure due to an increase in dimensionality may not be as great as generally reported in the literature if the proper statistical formulation of the problem is used. If this is the case, the analysis of one-dimensional systems will be as useful as the analysis of two- and three-dimensional systems for the stochastic solving of flow through porous media problems. It is obvious that further research is needed to confirm this postulation.

3. Although mobility ratio was taken into consideration in this study, it was treated as a deterministic variable. Because this parameter is not known exactly in real-life problems, it would be more appropriately treated as a random variable. Other properties, such as rock-fluid and fluid-fluid interaction properties, are not exactly known either. Consequently, it may be more appropriate to treat them as random variables as well. Investigation of the effect of uncertainty of these parameters is useful because if it is known, time and effort can be directed towards the measurement of the parameters whose uncertainty have more influence on the final results, and less towards the measurement of the parameters whose uncertainties have less influence on the final results.
4. If complete data are available, they can be used to test the usefulness of the stochastic methods by employing the small scale and large scale applications suggested

in Chapters VII and VIII.

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APPENDIX A

The effective permeabilities for one-dimensional, parallel and serial systems are derived in this appendix. The major assumption is that the permeability is log-normally distributed. It will be assumed also that the distributions of permeabilities at all points are the same with mean, μ_Y, and standard deviation, σ_Y, where Y is equal to ln(k).

A. Effective permeability for one-dimensional, parallel systems

For a realization of a parallel system, shown in Figure A-1, the permeability for the system, k^*, can be written as

$$k^* = \frac{1}{H} \sum_{i=1}^{n} k_i \Delta z_i , \tag{A-1}$$

where Δz_i is the thickness of layer i, H is the thickness of the system, and k_i is the permeability of layer i. Let

$$b_i = \frac{\Delta z_i}{H} . \tag{A-2}$$

Eq. (A-1) becomes

$$k^* = \sum_{i=1}^{n} b_i k_i \tag{A-3}$$

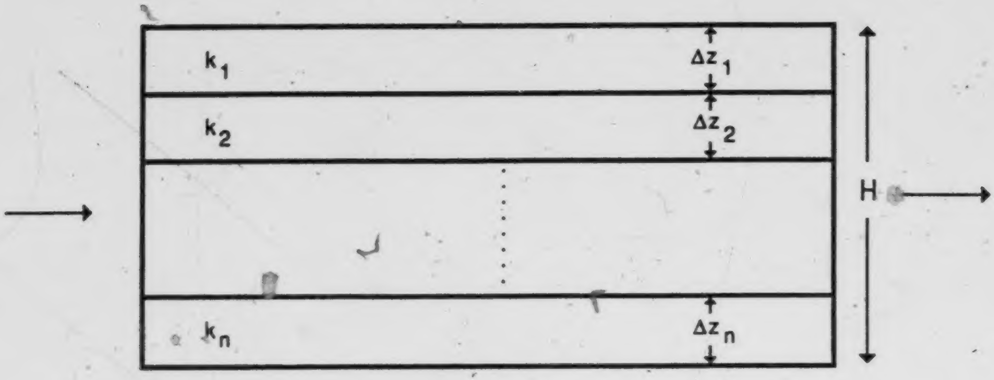

Figure A-1 Flow in a parallel system.

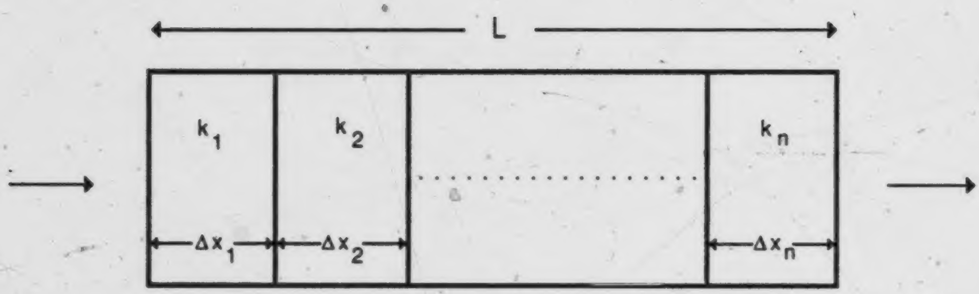

Figure A-2 Flow in a serial system.

The effective permeability may be defined as the expected value of k^* (evaluated from an infinitely large number of realizations) or

$$k_{eff} = E\{k^*\}. \tag{A-4}$$

It should be noted that when the effective permeability is defined in this manner, it is implicitly assumed that the area, viscosity and pressure gradient in Darcy's equation are determistic quantities while the flow rate and k^* are random variables. These assumptions are logical because, in the parallel system, the pressure drops across any layer are equal to one another and are kept constant. The variable that is changed due to a change in permeability is the flow rate. Therefore, flow rate should be treated as a random variable to be consistent with permeability, which is treated as a random variable. For each realization, Darcy's equation can be written as

$$q = - \frac{k^* A}{\mu} \frac{\Delta p}{\Delta L}, \tag{A-5}$$

where Δp is pressure difference over the distance ΔL. Taking the expectation (in accordance with the assumptions above),

$$E\{q\} = - E\{k^*\} \frac{A}{\mu} \frac{\Delta p}{\Delta L}. \tag{A-6}$$

Therefore, the effective permeability is defined as given in

Eq. (A-4). Using Eq. (A-3) in Eq. (A-4),

$$k_{eff} = E\{\sum_{i=1}^{n} b_i k_i\}, \qquad \text{(A-7)}$$

or

$$k_{eff} = \sum_{i=1}^{n} b_i \; E\{k_i\}. \qquad \text{(A-8)}$$

Because the k_i's are log-normally distributed, the expectation of k_i can be expressed as,[157]

$$E\{k_i\} = \exp(\mu_Y + \tfrac{1}{2}\sigma_Y^2),$$

where μ_Y and σ_Y are the mean and standard deviation of Y or ln(k), respectively. Therefore, Eq. (A-8) can be written as

$$k_{eff} = \exp(\mu_Y + \tfrac{1}{2}\sigma_Y^2), \qquad \text{(A-9)}$$

because $\sum_{i=1}^{n} b_i = 1$, or

$$k_{eff} = k_G \exp(\tfrac{1}{2}\sigma_Y^2), \qquad \text{(A-10)}$$

where $k_G = \exp(\mu_Y)$. Eq. (A-10) is the same as Eq. (II-4) in the text.

B. Effective permeability for one-dimensional, serial systems

The permeability for a realization of a serial system can be written in terms of input permeability as (see Figure A-2)

$$\frac{1}{k^*} = \frac{1}{L}\sum_{i=1}^{n} \frac{\Delta x_i}{k_i}, \qquad \text{(A-11)}$$

where Δx_i is the length of layer i, and L is the length of the system. Let

$$a_i = \frac{\Delta x_i}{L}, \qquad \text{(A-12)}$$

then,

$$\frac{1}{k^*} = \sum_{i=1}^{n} \frac{a_i}{k_i}, \qquad \text{(A-13)}$$

For single-phase flow in a serial system, Darcy's equation, for each realization, can be written as

$$q\,\frac{1}{k^*} = -\frac{A}{\mu}\,\frac{\Delta p}{\Delta L}. \qquad \text{(A-14)}$$

For flow in each realization, q is a constant; therefore, q, μ, and A should be treated as deterministic quantities while k^* and pressure gradient are treated as random variables. Taking the expectation of Eq. (A-14)

$$q\,E\{\frac{1}{k^*}\} = -\frac{A}{\mu}\,E\{\frac{\Delta p}{\Delta L}\},$$

or

$$q = -\frac{A}{\mu}\frac{1}{E\{1/k^*\}}E\{\frac{\Delta p}{\Delta L}\}. \qquad (A-15)$$

Therefore, for a serial system, the effective permeability can be defined by

$$k_{eff} = \frac{1}{E\{1/k^*\}}. \qquad (A-16)$$

By taking the expectation, Eq. (A-13) becomes

$$E\{\frac{1}{k^*}\} = \sum_{i=1}^{n} a_i E\{\frac{1}{k_i}\}. \qquad (A-17)$$

Each $E\{1/k_i\}$ can be evaluated as follows (note that each k_i has the same distribution with mean (of log-permeability), μ_Y, and standard deviation, σ_Y.

Because Y is normally distributed, the density function of Y is

$$f_Y(y) = \frac{1}{(2\pi)^{1/2}\sigma_Y}\exp\left[-\frac{1}{2}(\frac{y-\mu_Y}{\sigma_Y})^2\right]. \qquad (A-18)$$

By definition, $E\{1/k_i\}$ can be written as

$$E\{\frac{1}{k_i}\} = E\{\exp(-Y_i)\}$$

$$= \int_{-\infty}^{\infty}\exp(-Y_i)f_{Y_i}(Y_i)dY_i$$

$$= \int_{-\infty}^{\infty} \exp(-Y_i) \frac{1}{(2\pi)^{1/2}\sigma_Y} \exp\left[-\frac{1}{2}\left(\frac{Y_i-\mu_Y}{\sigma_Y}\right)^2\right] dY_i$$

$$= \left[\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}\sigma_Y} \exp\left\{-\frac{1}{2}\left[\frac{Y_i-(\mu_Y-\sigma_Y^2)}{\sigma_Y}\right]^2\right\} dY_i\right] \exp(-\mu_Y+\frac{1}{2}\sigma_Y^2)$$

or

$$E\{\frac{1}{k_i}\} = \exp(-\mu_Y+\frac{1}{2}\sigma_Y^2), \qquad \text{(A-19)}$$

because the integral in the brackets is the total unit area of the Gaussian density function $N(\mu_Y-\sigma_Y^2,\sigma_Y)$. Substituting Eq. (A-19) into Eq. (A-17),

$$E\{\frac{1}{k^*}\} = \exp(-\mu_Y+\frac{1}{2}\sigma_Y^2), \qquad \text{(A-20)}$$

because $\sum_{i=1}^{n} a_i = 1$, Therefore,

$$k_{eff} = \exp(\mu_Y-\frac{1}{2}\sigma_Y^2), \qquad \text{(A-21)}$$

or

$$k_{eff} = k_G\exp(-\frac{1}{2}\sigma_Y^2), \qquad \text{(A-22)}$$

This is the same as Eq. (II-3) in the text.

APPENDIX B

In this appendix the covariance functions for one-, two-, and three-dimensional systems are derived from the expression in Eq. (IV-7) in the text which is rewritten here

$$C_{x,p}(h) = \frac{1}{2}\left(\frac{h}{\alpha}\right)^{2-p/2}K_{2-p/2}(\alpha h), \tag{B-1}$$

A. Covariance function for one-dimensional systems

Here p = 1 and Eq. (B-1) becomes

$$C_{x,1}(h) = \frac{1}{2}\left(\frac{h}{\alpha}\right)^{3/2}K_{3/2}(\alpha h), \tag{B-2}$$

From Ref. 158 [p. 444, Eq. (10.2.17)]

$$K_{3/2}(\alpha h) = \left(\frac{\pi}{2}\right)^{1/2}(\alpha h)^{-1/2}e^{-\alpha h}\left(1+\frac{1}{\alpha h}\right).$$

Therefore,

$$C_{x,1}(h) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{\alpha^3}(1+\alpha h)e^{-\alpha h}, \tag{B-3}$$

and

$$C_{x,1}(0) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{\alpha^3} \tag{B-4}$$

Then,

$$\rho_{x,1}(h) = \frac{C_{x,1}(h)}{C_{x,1}(0)} = (1+\alpha h)e^{-\alpha h}. \tag{B-5}$$

B. Covariance function for two-dimensional systems

For p = 2, Eq. (B-1) becomes

$$C_{x,2}(h) = \frac{1}{2}\frac{1}{\alpha^2}\alpha h K_1(\alpha h). \tag{B-6}$$

From Ref. 158 [p. 375, Eq. (9.6.9)], as $\alpha h \to 0$,

$$K_1(\alpha h) \approx \frac{1}{2}\Gamma(1)(\tfrac{1}{2}\alpha h)^{-1} = (\alpha h)^{-1}.$$

Therefore,

$$C_{x,2}(h) \approx \frac{1}{2\alpha^2} \qquad \text{as } h \to 0, \tag{B-7}$$

and

$$C_{x,2}(0) = \frac{1}{2\alpha^2} \qquad \text{as } h \to 0, \tag{B-8}$$

Then,

$$\rho_{x,2}(h) = \frac{C_{x,2}(h)}{C_{x,2}(0)} = \alpha h K_1(\alpha h). \tag{B-9}$$

C. Covariance function for three-dimensional systems

For p = 3, Eq. (B-1) can be written as

$$C_{x,3}(h) = \frac{1}{2}\frac{1}{\alpha}(\alpha h)^{1/2}K_{1/2}(\alpha h). \tag{B-10}$$

Using expression (10.2.17) in Ref. 158 (p. 444), Eq. (B-10) becomes

$$C_{x,3}(h) = \frac{1}{2} \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{\alpha} \exp(-\alpha h), \qquad \text{(B-11)}$$

and

$$C_{x,3}(0) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} \frac{1}{\alpha}. \qquad \text{(B-12)}$$

Then,

$$\rho_{x,3}(h) = \frac{C_{x,3}(h)}{C_{x,3}(0)} = \exp(-\alpha h). \qquad \text{(B-13)}$$

APPENDIX C

The variances of oil pressure for one-, two-, and three-dimensional systems are derived in this appendix. Only the functional form of variance of oil pressure for the three-demensional system is obtained. The variance of oil pressure for one- and two-dimensional systems cannot be obtained because of a singularity problem.

A. Variance of oil pressure for one-dimensional systems

The relationship between the spectral density functions of p and f, expressed in Eq. (IV-46) in the text, is rewritten here as

$$S_{pp}(\omega) = \frac{J^2}{\omega^2} S_{ff}(\omega), \qquad \text{(C-1)}$$

To obtain the variance of the oil pressure (or of the perturbation part of the oil pressure, to be strictly correct) the autocorrelation function of f is needed. Then, the autocorrelation function of p can be obtained by using Eqs. (IV-44), (IV-46) or (C-1), and (IV-43), respectively.

Using the exponential form for the autocorrelation function of f, this function can be written as

$$C_{ff}(h) = \sigma_f^2 e^{-\alpha|h|}, \qquad \text{(C-2)}$$

[see Eq. (IV-11)]. Now using Eq. (IV-44),

$$S_{tt}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega h}\sigma_t^2 e^{-\alpha|h|}dh, \qquad \text{(C-3)}$$

Note that |h| is used here because h in the exponential autocorrelation function must always be positive while h in the exponential term used for the Fourier transform can be either positive or negative. It can be shown that the integral in Eq. (C-3) can be written as[52,159]

$$S_{tt}(\omega) = \frac{\sigma_t^2\alpha}{\pi(\alpha^2+\omega^2)}. \qquad \text{(C-4)}$$

Therefore, using Eqs. (C-1) and (IV-43), the covariance of p can be written as

$$C_{pp}(h) = \frac{J^2\sigma_t^2\alpha}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega h}}{\omega^2(\alpha^2+\omega^2)}\, d\omega. \qquad \text{(C-5)}$$

By definition, the variance of p can be expressed as[52,56]

$$\sigma_p^2 = C_{pp}(0) = \frac{J^2\sigma_t^2\alpha}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2(\alpha^2+\omega^2)}. \qquad \text{(C-6)}$$

The integral in Eq. (C-6) has a singularity at $\omega = 0$ and the variance of p cannot be obtained. It is suspected that the singularity problem may have arisen because of the assumption that p is statistically homogeneous. In fact, if p is not (second-order) statistically homogeneous, p might not have the spectral representation, Eq. (IV-35), and the spectral techniques cannot be applied.

B. Variance of oil pressure for two-dimensional systems

The spectral density function of f, using the exponential autocorrelation function, can be expressed as

$$S_{ff}(\underline{\omega}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} e^{-i\underline{\omega}\cdot\underline{h}}\sigma_f^2 e^{-\alpha h}d\underline{h}, \qquad \text{(C-7)}$$

where h is the magnitude of $\underline{h}$ and is always positive. If f is statistically homogeneous and statistically isotropic, $\underline{h}$ can be transformed into polar coordinates having a radial axis in the direction of $\underline{\omega}$. Consequently, Eq. (C-7) can be written (after some manipulation) as

$$S_{ff}(\underline{\omega}) = \frac{\sigma_f^2}{2\pi}\int_0^{\infty} e^{-\alpha h} h J_0(\omega h)dh, \qquad \text{(C-8)}$$

Using formulae in Ref. 155 (p. 711 and p. 1025), Eq. (C-8) becomes

$$S_{ff}(\underline{\omega}) = \frac{\sigma_f^2}{2\pi}\,\alpha\,(\omega^2+\alpha^2)^{-3/2}. \qquad \text{(C-9)}$$

Therefore, the spectral density function for p, [Eq. (IV-47)] is

$$S_{pp}(\underline{\omega}) = \frac{J_1^2\omega_1^2}{\omega^4}\frac{\sigma_f^2}{2\pi}\,\alpha\,(\omega^2+\alpha^2)^{-3/2}, \qquad \text{(C-10)}$$

and

$$C_{pp}(\underline{h}) = \frac{J_1^2\sigma_f^2\alpha}{2\pi}\iint_{-\infty}^{\infty} e^{i\underline{\omega}\cdot\underline{h}}\frac{\omega_1^2}{\omega^4}(\omega^2+\alpha^2)^{-3/2}\,d\underline{\omega}. \qquad \text{(C-11)}$$

Using the definition of variance, σ_p^2 can be written as

$$\sigma_p^2 = C_{pp}(0) = \frac{J_1^2\sigma_f^2\alpha}{2\pi} \iint_{-\infty}^{\infty} \frac{\omega_1^2}{\omega^4} (\omega^2+\alpha^2)^{-3/2} \, d\omega_1 d\omega_2 . \qquad \text{(C-12)}$$

Transforming ω_1 and ω_2 into polar coordinates, Eq. (C-12) becomes

$$\sigma_p^2 = \frac{J_1^2\sigma_f^2\alpha}{2\pi} \int_0^{2\pi}\int_0^{\infty} \frac{\cos^2\theta}{\omega} (\omega^2+\alpha^2)^{-3/2} \, d\omega d\theta$$

$$\sigma_p^2 = \frac{J_1^2\sigma_f^2\alpha}{2} \int_0^{\infty} \frac{d\omega}{\omega(\omega^2+\alpha^2)^{3/2}} . \qquad \text{(C-13)}$$

The integral in Eq. (C-13) has a singularity at $\omega = 0$. The singularity problem may have arisen because of the same cause as in the case of a one-dimensional system.

C. Variance of oil pressure for three-dimensional systems

Using the same procedure as in the case of a one-dimensional system, the spectral density function of f can be written as

$$S_{ff}(\underline{\omega}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-i\underline{\omega}\cdot\underline{h}} \sigma_f^2 e^{-\alpha h} d\underline{h}, \qquad \text{(C-14)}$$

where h is the magnitude of $\underline{h}$ and is always positive. By the assumption that f is statistically homogeneous and statistically isotropic, $\underline{h}$ can be transformed into spherical coordinates having a polar axis in the direction of $\underline{\omega}$. If this is done, Eq. (C-14) can be written as[52]

$$S_{ff}(\underline{\omega}) = \frac{\sigma_f^2}{2\pi^2\omega}\int_0^\infty \sin(\omega h)he^{-\alpha h}dh, \qquad \text{(C-15)}$$

where ω is the magnitude of $\underline{\omega}$. From Ref. 160 (p. 490), Eq. (C-15) can be written as

$$S_{ff}(\underline{\omega}) = \frac{\sigma_f^2}{\pi^2}\,\frac{\alpha}{(\omega^2+\alpha^2)^2}. \qquad \text{(C-16)}$$

Then, using Eq. (IV-47),

$$S_{pp}(\underline{\omega}) = \frac{J_1^2\omega_1^2}{\omega^4}\,\frac{\sigma_f^2}{\pi^2}\,\frac{\alpha}{(\omega^2+\alpha^2)^2}, \qquad \text{(C-17)}$$

and

$$C_{pp}(\underline{h}) = \frac{J_1^2\sigma_f^2\alpha}{\pi^2}\iiint_{-\infty}^{\infty} e^{i\underline{\omega}\cdot\underline{h}}\frac{\omega_1^2}{\omega^4(\omega^2+\alpha^2)^2}\,d\underline{\omega}. \qquad \text{(C-18)}$$

Again, by definition of variance,

$$\sigma_p^2 = C_{pp}(0) = \frac{J_1^2\sigma_f^2\alpha}{\pi^2}\iiint_{-\infty}^{\infty}\frac{\omega_1^2}{\omega^4(\omega^2+\alpha^2)^2}\,d\omega_1 d\omega_2 d\omega_3. \qquad \text{(C-19)}$$

Transforming ω_1, ω_2, ω_3 into spherical coordinates, Eq. (C-19) becomes

$$\begin{aligned}\sigma_p^2 &= \frac{J_1^2\sigma_f^2\alpha}{\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\cos^2\theta_1\,\sin\theta_1}{(\omega^2+\alpha^2)^2}\,d\omega d\theta_1 d\theta_2\\ &= \frac{2J_1^2\sigma_f^2\alpha}{\pi}\int_0^{\infty}\int_0^{\pi}\cos^2\theta_1\,\sin\theta_1\,d\theta_1\,\frac{d\omega}{(\omega^2+\alpha^2)^2}\\ &= \frac{4}{3}\,\frac{J_1^2\sigma_f^2\alpha}{\pi}\int_0^{\infty}\frac{d\omega}{(\omega^2+\alpha^2)^2}\end{aligned}$$

and, finally,

$$\sigma_p^2 = \frac{J_1^2\sigma_\ell^2}{3\alpha^2}. \tag{C-20}$$

or

$$\sigma_p^2 = \frac{1}{3} J_1^2\sigma_\ell^2\lambda_\ell^2. \tag{C-21}$$

which is Eq. (IV-49) in the text.

APPENDIX D

This appendix presents the derivation of the expectation of estimates of variance and autocovariance for processes which have an exponential correlation function.

The autocovariance can be written as

$$C(h) = \sigma^2 \exp(-\frac{h}{\lambda}), \tag{D-1}$$

and

$$\rho(h) = \frac{C(h)}{\sigma^2} = \exp(-\frac{h}{\lambda}). \tag{D-2}$$

The estimates of mean, variance, and autocovariance are

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \tag{D-3}$$

$$s^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \overline{x})^2, \tag{D-4}$$

$$c(k) = \frac{1}{(n-k)} \sum_{i=1}^{n-k} (x_i - \overline{x})(x_{i+k} - \overline{x}) \tag{D-5}$$

Let (for equal block size)

$$\rho_1 = \exp(-\frac{h_1}{\lambda}) = \rho,$$

then

$$\rho_2 = \exp(-\frac{h_2}{\lambda}) = \exp(-\frac{2h_1}{\lambda}) = \rho^2,$$

$$\rho_3 = \exp(-\frac{h_3}{\lambda}) = \exp\ (-\frac{3h_1}{\lambda}) = \rho^3,$$

$$\vdots$$

$$\rho_k = \exp(-\frac{h_k}{\lambda}) = \exp\ (-\frac{kh_1}{\lambda}) = \rho^k, \tag{D-6}$$

Taking the expectation of Eq. (D-3),

$$E\{\overline{x}\} = \frac{1}{n}\sum_{i=1}^{n} E\{x_i\} = \frac{1}{n}n\mu = \mu. \tag{D-7}$$

Therefore, $\overline{x}$ is an unbiased estimate of the mean, μ. Taking the expectation of Eq. (D-4),

$$\begin{aligned} E\{s^2\} &= \frac{1}{(n-1)}E\{\sum_{i=1}^{n}(x_i - \overline{x})^2\} \\ &= \frac{1}{(n-1)}E\{\sum_{i=1}^{n}(x_i^2 - 2x_i\overline{x} + \overline{x}^2)\} \end{aligned}$$

$$E\{s^2\} = \frac{1}{(n-1)}\left[\sum_{i=1}^{n}E\{x_i^2\} - 2E\{\sum_{i=1}^{n}x_i\overline{x}\} + \sum_{i=1}^{n}E\{\overline{x}^2\}\right]. \tag{D-8}$$

But

$$\overline{x}^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}x_i x_j = \frac{1}{n^2}\sum_{i=1}^{n}x_i^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}x_i x_j,$$

then,

$$\begin{aligned} E\{\overline{x}^2\} = \frac{1}{n^2}\sum_{i=1}^{n}E\{x_i^2\} + \frac{2}{n^2}\Big[&(n-1)E\{x_i x_{i+1}\} + (n-2)E\{x_i x_{i+2}\} \\ &+ \ldots\ldots +2E\{x_i x_{i+(n-2)}\} + E\{x_i x_{i+(n-1)}\}\Big] \end{aligned}$$

$$= \frac{1}{n^2}n(\sigma^2+\mu^2) + \frac{2}{n^2}\Big[(n-1)(\sigma^2\rho_1+\mu^2) + (n-2)(\sigma^2\rho_2+\mu^2)$$

$$+ \;.....\; + 2(\sigma^2\rho_{n-2}+\mu^2) + (\sigma^2\rho_{n-1}+\mu^2)\Big]$$

$$= \frac{1}{n}(\sigma^2+\mu^2) + \frac{2}{n^2}\Big[(n-1)(\sigma^2\rho+\mu^2) + (n-2)(\sigma^2\rho^2+\mu^2)$$

$$+ \;.....\; + 2(\sigma^2\rho^{n-2}+\mu^2) + (\sigma^2\rho^{n-1}+\mu^2)\Big]$$

$$= \frac{\sigma^2}{n} + \frac{\mu^2}{n} + \frac{2\sigma^2}{n^2}\Big[(n-1)\rho + (n-2)\rho^2 + \;.....\; + 2\rho^{n-2}$$

$$+ \rho^{n-1}\Big] + \frac{2\mu^2}{n^2}\Big[(n-1) + (n-2) + \;.....\; + 2 + 1\Big]$$

$$= \frac{\sigma^2}{n} + \frac{\mu^2}{n} + \frac{2\sigma^2}{n^2}\left[\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\right] + \frac{(n-1)}{n}\mu^2$$

$$E\{\overline{x}^2\} = \mu^2 + \frac{\sigma^2}{n} + \frac{2\sigma^2}{n^2}\left[\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\right], \qquad \text{(D-9)}$$

and

$$x_i\overline{x} = x_i\,\frac{1}{n}\sum_{j=1}^{n} x_j,$$

or

$$\sum_{i=1}^{n} x_i\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i\sum_{j=1}^{n} x_j = \frac{1}{n}\Big[\sum_{i=1}^{n} x_i \text{[illegible]} \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} x_i x_j\Big],$$

then,

$$E\{\sum_{i=1}^{n} x_i\overline{x}\} = \frac{1}{n}\Big[\sum_{i=1}^{n} E\{x_i^2\} + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} E\{x_i x_j\}\Big]$$

$$= \frac{1}{n}\Big[n(\sigma^2+\mu^2) + 2\sigma^2\{\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\} + n(n-1)\mu^2)\Big]$$

$$E\{\sum_{i=1}^{n} x_i\bar{x}\} = n\mu^2 + \sigma^2 + \frac{2\sigma^2}{n}[\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}]. \qquad \text{(D-10)}$$

Substituting Eq. (D-9) and (D-10) into Eq. (D-8),

$$E\{s^2\} = \frac{1}{n-1}\Big[n(\sigma^2+\mu^2) - 2n\mu^2 - 2\sigma^2 - \frac{4\sigma^2}{n}\{\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\}$$

$$+ n\{\mu^2 + \frac{\sigma^2}{n} + \frac{2\sigma^2}{n^2}[\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}]\}\Big]$$

$$= \frac{1}{n-1}\Big[(n-1)\sigma^2 - \frac{2\sigma^2}{n}[\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}]\Big],$$

or

$$E\{s^2\} = \sigma^2\Big[1 - \frac{2}{n(n-1)}\{\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\}\Big], \qquad \text{(D-11)}$$

which is Eq. (V-7) in the text.

Taking the expectation of Eq. (D-5),

$$E\{c(k)\} = \frac{1}{n-k}\,E\{\sum_{i=1}^{n-k}(x_i x_{i+k} - x_i\bar{x} - x_{i+k}\bar{x} + \bar{x}^2)\}$$

$$E\{c(k)\} = \frac{1}{n-k}\,\Big[\sum_{i=1}^{n-k}E\{x_i x_{i+k}\} - E\{\sum_{i=1}^{n-k}x_i\bar{x} + \sum_{i=1}^{n-k}x_{i+k}\bar{x}\}$$

$$+ \sum_{i=1}^{n-k}E\{\bar{x}^2\}\Big]. \qquad \text{(D-12)}$$

The middle terms on the right hand side of Eq. (D-12) can be considered for 3 different cases as follows:

I. When $k < n-k$ or $k < n/2$,

$$E\{\sum_{i=1}^{n-k} x_i\overline{x} + \sum_{i=1}^{n-k} x_{i+k}\overline{x}\} = E\{\overline{x}[\sum_{i=1}^{n-k} x_i + \sum_{i=k+1}^{n} x_i]\}$$

$$= E\{\overline{x}[x_1+x_2+\ldots\ldots+X_n]$$

$$+ \overline{x}[x_{k+1}+x_{k+2}+\ldots\ldots+x_{n-k}]\}$$

$$= E\{\overline{x}\sum_{i=1}^{n} x_i + \overline{x}\sum_{i=k+1}^{n-k} x_i\}. \qquad \text{(D-13)}$$

The last term in Eq. (D-13) can be expanded as

$$E\{\overline{x}\sum_{i=k+1}^{n-k} x_i\} = E\{\frac{1}{n}\sum_{j=1}^{n} x_j \sum_{i=k+1}^{n-k} x_i\}$$

$$= E\{\frac{1}{n}\Big[\sum_{j=1}^{k} x_j + \sum_{j=k+1}^{n-k} x_j + \sum_{j=n-k+1}^{n} x_j\Big] \sum_{i=k+1}^{n-k} x_i\}$$

$$= E\{\frac{1}{n}\sum_{j=1}^{k} x_j\sum_{i=k+1}^{n-k} x_i + \frac{1}{n}\sum_{j=k+1}^{n-k} x_j\sum_{i=k+1}^{n-k} x_i$$

$$+ \frac{1}{n}\sum_{j=n-k+1}^{n} x_j\sum_{i=k+1}^{n-k} x_i\}, \qquad \text{(D-14)}$$

and

$$E\{\sum_{j=1}^{k} x_j\sum_{i=k+1}^{n-k} x_i\} = E\{x_1(x_{k+1} + x_{k+2} + \ldots\ldots + x_{n-k})$$

$$+x_2(x_{k+1} + x_{k+2} + \ldots\ldots + x_{n-k})$$

$$\vdots$$

$$+x_k(x_{k+1} + x_{k+2} + \ldots\ldots + x_{n-k})\}$$

$$= E\{x_1x_{k+1}\} + E\{x_1x_{k+2}\} + \ldots\ldots + E\{x_1x_{n-k}\}$$

$$+ E\{x_2 x_{k+1}\} + E\{x_2 x_{k+2}\} + \ldots\ldots + E\{x_2 x_{n-k}\}$$

$$\vdots$$

$$+ E\{x_k x_{k+1}\} + E\{x_k x_{k+2}\} + \ldots\ldots + E\{x_k x_{n-k}\}$$

$$= (\sigma^2\rho^k+\mu^2)+(\sigma^2\rho^{k+1}+\mu^2)+\ldots\ldots+(\sigma^2\rho^{n-k-1}+\mu^2)$$

$$+(\sigma^2\rho^{k-1}+\mu^2)+(\sigma^2\rho^k+\mu^2)+\ldots\ldots+(\sigma^2\rho^{n-k-2}+\mu^2)$$

$$\vdots$$

$$+(\sigma^2\rho+\mu^2)+(\sigma^2\rho^2+\mu^2)+\ldots\ldots+(\sigma^2\rho^{n-k-k}+\mu^2)$$

$$= k(n-2k)\mu^2+\sigma^2\Big[\{\rho+\rho^2+\ldots\ldots+\rho^k\}+\{\rho^2+\rho^3+\ldots\ldots+\rho^{k+1}\}$$

$$+\ldots\ldots+\{\rho^{n-k-k}+\rho^{n-k-(k-1)}+\ldots\ldots+\rho^{n-k-1}\}\Big]$$

$$= k(n-2k)\mu^2+\sigma^2\Big[\frac{\rho-\rho^{k+1}}{(1-\rho)} + \frac{\rho^2-\rho^{k+2}}{(1-\rho)} + \ldots\ldots$$

$$+ \frac{\rho^{n-k-k}-\rho^{n-k}}{(1-\rho)}\Big]$$

$$= k(n-2k)\mu^2+\frac{\sigma^2}{(1-\rho)}\Big[\{\rho+\rho^2+\ldots\ldots+\rho^{n-2k}\}$$

$$-\{\rho^{k+1}+\rho^{k+2}+\ldots\ldots+\rho^{n-k}\}\Big]$$

$$= k(n-2k)\mu^2+\frac{\sigma^2}{(1-\rho)}\Big[\frac{\rho-\rho^{n-2k+1}}{(1-\rho)} - \frac{\rho^{k+1}-\rho^{n-k+1}}{(1-\rho)}\Big]$$

$$= k(n-2k)\mu^2+\frac{\sigma^2}{(1-\rho)^2}\Big[\rho(1-\rho^{n-2k})-\rho^{k+1}(1-\rho^{n-2k})\Big]$$

$$E\{\sum_{j=1}^{k} x_j \sum_{i=k+1}^{n-k} x_i\} = k(n-2k)\mu^2 + \frac{\sigma^2}{(1-\rho)^2}(1-\rho^{n-2k})\rho(1-\rho^k). \qquad \text{(D-15)}$$

Similarly, we have

$$E\{\sum_{j=n-k+1}^{n} x_j \sum_{i=k+1}^{n-k} x_i\} = k(n-2k)\mu^2 + \frac{\sigma^2}{(1-\rho)^2}(1-\rho^{n-2k})\rho(1-\rho^k), \qquad \text{(D-16)}$$

and [see Eq. (D-10)]

$$E\{\sum_{i=k+1}^{n-k}\sum_{j=k+1}^{n-k} x_i x_j\} = (n-2k)\sigma^2 + (n-2k)^2\mu^2$$

$$+ 2\sigma^2\left[\frac{(n-2k)\rho(1-\rho)-\rho(1-\rho^{n-2k})}{(1-\rho)^2}\right]. \qquad \text{(D-17)}$$

Then Eq. (D-14) becomes

$$E\{\overline{x}\sum_{i=k+1}^{n-k} x_i\} = \frac{2k(n-2k)}{n}\mu^2 + \frac{2\sigma^2}{n(1-\rho)^2}(1-\rho^{n-2k})\rho(1-\rho^k) + \frac{(n-2k)}{n}\sigma^2$$

$$+ \frac{(n-2k)^2}{n}\mu^2 + \frac{2\sigma^2}{n}\left[\frac{(n-2k)\rho(1-\rho)-\rho(1-\rho^{n-2k})}{(1-\rho)^2}\right]$$

and Eq. (D-13) becomes

$$E\{\sum_{i=1}^{n-k} x_i\overline{x} + \sum_{i=1}^{n-k} x_{i+k}\overline{x}\} = n\mu^2 + \sigma^2 + \frac{2\sigma^2}{n}\left[\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\right]$$

$$+ \frac{2k(n-2k)}{n}\mu^2 + \frac{2\sigma^2}{n(1-\rho)^2}(1-\rho^{n-2k})\rho(1-\rho^k)$$

$$+ \frac{(n-2k)}{n}\sigma^2 + \frac{(n-2k)^2}{n}\mu^2$$

$$+ \frac{2\sigma^2}{n}\left[\frac{(n-2k)\rho(1-\rho)-\rho(1-\rho^{n-2k})}{(1-\rho)^2}\right]. \qquad \text{(D-18)}$$

Substituting Eq. (D-18) into Eq. (D-12),

$$\begin{aligned}
E\{c(k)\} = \frac{1}{n-k}\Bigg[& (n-k)(\sigma^2\rho^k+\mu^2) - n\mu^2 - \sigma^2 \\
& - \frac{2\sigma^2}{n}\left\{\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\right\} - \frac{2k(n-2k)}{n}\mu^2 \\
& - \frac{2\sigma^2(1-\rho^{n-2k})\rho(1-\rho^k)}{n(1-\rho)^2} - \frac{(n-2k)}{n}\sigma^2 \\
& - \frac{(n-2k)^2}{n}\mu^2 - \frac{2\sigma^2}{n^2}\left\{\frac{(n-2k)\rho(1-\rho)-\rho(1-\rho^{n-2k})}{(1-\rho)^2}\right\} \\
& + (n-k)\left\{\mu^2 + \frac{\sigma^2}{n} + \frac{2\sigma^2}{n^2}\left[\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\right]\right\}\Bigg].
\end{aligned}$$

Rearranging,

$$\begin{aligned}
E\{c(k)\} = {} & \mu^2\left[\frac{2n^2-2nk-n^2-2nk+4k^2-n^2+4nk-4k^2}{n(n-k)}\right] \\
& + \sigma^2\Bigg[\rho^k - \frac{1}{n} - \frac{2k}{n^2(n-k)}\left\{\frac{n\rho(1-\rho)-\rho(1-\rho^n)}{(1-\rho)^2}\right\} \\
& \qquad - \frac{2}{n(n-k)}\left\{\frac{(n-2k)\rho(1-\rho)-\rho(1-\rho^{n-2k})}{(1-\rho)^2}\right\} \\
& \qquad - \frac{2(1-\rho^{n-2k})\rho(1-\rho^k)}{n(n-k)(1-\rho)^2}\Bigg]
\end{aligned}$$

$$E\{c(k)\} = \sigma^2\rho^k\left[1 - \frac{1}{\rho^k}\left\{\frac{1}{n} + \frac{2\rho}{n(1-\rho)}\right.\right.$$

$$- \frac{2\rho[k(1-\rho^{n})+n\rho^{k}(1-\rho^{n-2k})]}{n^{2}(n-k)(1-\rho)^{2}}\}\Big] \qquad \text{(D-19)}$$

for the case that k < n/2.

II. When k > n-k or k > n/2,

$$E\{\sum_{i=1}^{n-k} x_i\overline{x} + \sum_{i=1}^{n-k} x_{i+k}\overline{x}\} = E\{\overline{x}[\sum_{i=1}^{n-k} x_i + \sum_{i=k+1}^{n} x_i]\}$$

$$= E\{\overline{x}[x_1+x_2+....+X_n]$$

$$- \overline{x}[x_{n-k+1}+x_{n-k+2}+....+x_k]\}$$

$$= E\{\overline{x}\sum_{i=1}^{n} x_i - \overline{x}\sum_{i=n-k+1}^{k} x_i\} \qquad \text{(D-20)}$$

Proceed as before, and finally

$$E\{c(k)\} = \sigma^2\rho^k\Big[1 - \frac{1}{\rho^k}\{\frac{1}{n} + \frac{2\rho}{n(1-\rho)}$$

$$- \frac{2\rho[k(1-\rho^{n})-n\rho^{n-k}(1-\rho^{2k-n})]}{n^{2}(n-k)(1-\rho)^{2}}\}\Big] \qquad \text{(D-21)}$$

for the case that k > n/2.

III. When k = n-k or k = n/2,

$$E\{\sum_{i=1}^{n-k} x_i\overline{x} + \sum_{i=1}^{n-k} x_{i+k}\overline{x}\} = E\{\overline{x}[\sum_{i=1}^{n-k} x_i + \sum_{i=k+1}^{n} x_i]\}$$

$$= E\{\overline{x}[\sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} x_i]\}$$

$$= E\{\overline{x}\sum_{i=1}^{n} x_i\}. \tag{D-22}$$

Substituting Eq. (D-22) into Eq. (D-12) and rearranging,

$$E\{c(k)\} = \sigma^2\rho^k\left[1 - \frac{1}{n^2\rho^k}\{\frac{n(1-\rho^2)-2\rho(1-\rho^n)}{(1-\rho)^2}\}\right] \tag{D-23}$$

for the case that k = n/2.

APPENDIX E

```
C       PROGRAM NAME: VARGEN
C       PURPOSE: TO GENERATE UNCONDITIONAL VARIATES
C                   WHICH HAVE A NORMAL DISTRIBUTION AND
C                   A SPECIFIED COVARIANCE FUNCTION.
C                   USE GGNPM IN THE IMSL LIBRARY.
C
C       METHOD: USE METHOD OF THE LU DECOMPOSITION OF
C                 THE COVRIANCE MATRIX.(SEE ANDERSON, T. W.,
C                 AN INTRODUCTION TO MULTIVARIATE STATISTICAL
C                 ANALYSIS, 1958; SCHEUER, E. M. AND STOLLER,
C                 D. S., TECHNOMETRICS, VOL. 4, 1962, 278-281;
C                 AND DAVIS, M. W., MATH. GEOL., VOL. 19,
C                 NO. 2, 1987, 99-107).
C                 USE DISCRETE RELATIONSHIP BETWEEN COVARIANCE
C                 FUNCTION AND DISTANCE IN EVALUATING INTEGRAL
C                 SCALE.
C
        IMPLICIT REAL*8(A-H,O-V,X-Z)
        EXTERNAL COVFUN
        EXTERNAL IADD
        DIMENSION IHEDIN(80),AK0(125250),AK(125250)
        DIMENSION SQRTD(500),Z(500),COVT(500),ICOUNT(500)
        DIMENSION COVG(500),Z1(500)
        DIMENSION CO(500),SU(500),COT(500),SUT(500),CTT(500)
        DIMENSION IC(500),DT(500),COVTV(500),COVTC(500),TSTATC(500)
        DIMENSION BIASCO(40),BIASCT(40),DIFBC(40),SEC(40),DBSEC(40)
        REAL*4 W(500)
        INTEGER N
C
C       READ IN MEAN, STANDARD DEVIATION, AND INTEGRAL
C       SCALE (AS FRACTION OF TOTAL LENGTH)
C
        READ(5,2000)(IHEDIN(IH),IH=1,40)
  2000  FORMAT(40A2)
        READ(5,2010)AMEAN,STD,AINTSC
  2010  FORMAT(3D15.6)
        WRITE(6,2000)(IHEDIN(IH),IH=1,40)
        WRITE(6,2010)AMEAN,STD,AINTSC
C
C       READ IN NUMBER OF VARIATES NEEDED AND SIZE OF BLOCKS
C
        READ(5,2000)(IHEDIN(IH),IH=1,40)
        READ(5,2020)IOP,N,DELX
  2020  FORMAT(2I10,D15.6)
        WRITE(6,2000)(IHEDIN(IH),IH=1,40)
        WRITE(6,2020)IOP,N,DELX
        READ(5,2000)(IHEDIN(IH),IH=1,40)
        READ(5,2030)DSEED,ISIM
  2030  FORMAT(D15.6,I10)
        WRITE(6,2000)(IHEDIN(IH),IH=1,40)
        WRITE(6,2030)DSEED,ISIM
        IF(STD.EQ.0.0D0)GO TO 77
C
C       INITIALIZE COVARIANCE MATRIX
C
        NC=N*(N+1)/2
        DO 10 I=1,NC
        AK0(I)=0.0D0
        AK(I)=0.0D0
     10 CONTINUE
C
```

```
C      ASSIGNING ELEMENTS OF MODIFIED COVARIATE MATRIX
C
       AMEAN2=AMEAN*AMEAN
       VAR=STD*STD
       DO 20 I=1,N
       L=IADD(I,I)
       AK0(L)=VAR
       AK(L)=VAR
   20  CONTINUE
       FRDELX=1.0D0/FLOAT(N)
       NM1=N-1
       IF(AINTSC.EQ.0.0D0)GO TO 72
       DO 30 I=1,NM1
       IP1=I+1
       LL=0
       DO 30 J=IP1,N
       L=IADD(J,I)
       LL=LL+1
       AL=FLOAT(LL)
       AK0(L)=COVFUN(VAR,AL,FRDELX,AINTSC)
       AK(L)=AK0(L)
   30  CONTINUE
       WRITE(6,3000)
 3000  FORMAT(///T10,'COVARIANCE MATRIX'/)
       DO 40 I=1,N
       WRITE(6,3010)(AK0(IADD(I,J)),J=1,I)
   40  CONTINUE
 3010  FORMAT(8D15.6)
C
C      CALCULATE MODIFIED UPPER MATRIX
C
       NBW=N
       CALL UDUB(AK,N,NBW,IER)
       WRITE(6,3500)IER
 3500  FORMAT(/T20,5('*'),' IER = ',I3,' ',5('*')/)
C
C      CALCULATE LOWER MATRIX
C
       DO 60 I=1,N
       L=IADD(I,I)
       SQRTD(I)=DSQRT(1.0D0/AK(L))
   60  CONTINUE
C
       DO 70 I=1,N
       DO 70 J=1,I
       L=IADD(I,J)
       AK(L)=AK(L)*SQRTD(J)
   70  CONTINUE
C
C      START MONTE CARLO SIMULATION
C
   72  AMT=0.0D0
       VT=0.0D0
       AMEANT=0.0D0
       VARTM=0.0D0
       VART=0.0D0
       VARTV=0.0D0
       AINTT=0.0D0
       AINTV=0.0D0
       DO 75 I=1,N
       COT(I)=0.0D0
       SUT(I)=0.0D0
       COVT(I)=0.0D0
       COVTV(I)=0.0D0
   75  CONTINUE
```

```
C
   77  DO 130 II=1,ISIM
       IF(STD.EQ.0.0D0)GO TO 85
C
C      ASSIGNING ELEMINTS FOR VECTOR W
C
       CALL GGNPM(DSEED,N,W)
C
C      CALCULATE VECTOR Z
C
   85  DO 90 I=1,N
       Z(I)=0.0D0
   90  CONTINUE
       IF(STD.EQ.0.0D0)GO TO 108
       IF(AINTSC.EQ.0.0D0)GO TO 102
C
       DO 100 I=1,N
       DO 100 J=1,I
       Z(I)=Z(I)+AK(IADD(I,J))*W(J)
  100   CONTINUE
       GO TO 108
  102   DO 104 I=1,N
       Z(I)=STD*W(I)
  104   CONTINUE
C
C      CALCULATE Y (IN PLACE OF Z), THE VARIATES SOUGHT
C
  108   DO 110 I=1,N
       Z(I)=Z(I)+AMEAN
       Z1(I)=Z(I)
  110   CONTINUE
       IF(IOP.EQ.0)GO TO 115
       DO 112 I=1,N
       Z1(I)=DEXP(Z(I))
  112   CONTINUE
  115   WRITE(7,3025)(Z1(I),I=1,N)
 3025   FORMAT(5E15.6)
C
C      PRINT OUT REQUIRED VARIATES
C
       WRITE(6,3020)
 3020   FORMAT(/T10,'ARRAY OF VARIATE[illegible]
       WRITE(6,3010)(Z(I),I=1,N)
       IF(STD.EQ.0.0D0)GO TO 130
       IF(AINTSC.EQ.0.0D0)GO TO 116
C
       RHO=AK0(2)/AK0(1)
       RHOM1=1.0D0-RHO
       RHOM1S=RHOM1*RHOM1
       FLN=FLOAT(N)
       RHON1=1.0D0-RHO**N
       VARCOR=1.0D0-(2.0D0/(FLN*FLOAT(N-1)))*(FLN*RHO*RHOM1
      &           -RHO*RHON1)/RHOM1S
       FLNS=FLOAT(N*N)
       FIRT=1.0D0/FLN
       SECT=2.0D0*RHO/(FLN*RHOM1)
       THIT=2.0D0*RHO/(FLNS*RHOM1S)
       VARTC=AK0(1)*VARCOR
       BIASVT=VARTC-AK0(1)
       GO TO 117
  116   VARTC=VAR
C
C      TESTING THE VARIATES GENERATED
C
  117   CALL TEST(Z,AK0,AMEAN,STD,AINTSC,N,0,1,AMEANG,VARG,COVG,ICOUNT,
     &                 AINTG,XX,XX2,CO,SU,VARTC)
C
      AMT=AMT+XX
      VT=VT+XX2
      AMEANT=AMEANT+AMEANG
      VARTM=VARTM+AMEANG*AMEANG
      VART=VART+VARG
      VARTV=VARTV+VARG*VARG
      IF(AINTSC.EQ.0.0D0)GO TO 130
      AINTT=AINTT+AINTG
      AINTV=AINTV+AINTG*AINTG
      DO 120 I=1,N
      COT(I)=COT(I)+CO(I)
      S[illegible]I)=SUT(I)+SU(I)
      C[illegible](I)=COVT(I)+COVG(I)
      COVTV(I)=COVTV(I)+COVG(I)*COVG(I)
  120  CONTINUE
  130  CONTINUE
      IF(STD.EQ.0.0D0)GO TO 190
      FT=FLOAT(ISIM*N)
      VT=(VT-AMT*AMT/FT)/FLOAT(ISIM*N-1)
      AMT=AMT/FT
      ST=DSQRT(VT)
      FISIM=FLOAT(ISIM)
      VARTM=(VARTM-AMEANT*AMEANT/FISIM)/FLOAT(ISIM-1)
      AMEANT=AMEANT/FISIM
      VARTV=(VARTV-VART*VART/FISIM)/FLOAT(ISIM-1)
      VART=VART/FISIM
      STDT=DSQRT(VART)
      BIASMO=AMEANT-AMEAN
      BIASVO=VART-VAR
      DIFBM=BIASMO-0.0D0
      DIFBV=BIASVO-BIASVT
      SEM=DSQRT(VARTM/FISIM)
      SEV=DSQRT(VARTV/FISIM)
      DBSEM=2.0D0*SEM-DABS(DIFBM)
      DBSEV=2.0D0*SEV-DABS(DIFBV)
      TSTAT=DABS(VART-VARTC)/DSQRT(VARTV/FISIM)
      IF(AINTSC.EQ.0.0D0)GO TO 141
      AINTV=(AINTV-AINTT*AINTT/FISIM)/FLOAT(ISIM-1)
      AINTT=AINTT/FISIM
      DO 140 I=1,N
      IC(I)=ICOUNT(I)*ISIM
      CTT(I)=(COT(I)-AMT*SUT(I)+FLOAT(IC(I))*AMT*AMT)/FLOAT(IC(I))
      COVTV(I)=(COVTV(I)-COVT(I)*COVT(I)/FISIM)/FLOAT(ISIM-1)
      COVT(I)=COVT(I)/FISIM
      IM1=I-1
      IF(I.NE.1)GO TO 123
      COVTC(I)=VARTC
      COVT(I)=VART
      COVTV(I)=VARTV
      GO TO 567
  123  IF(IM1.GE.N-IM1)GO TO 234
      COVTC(I)=1.0D0-(FIRT+SECT-(THIT/FLOAT(N-IM1))
     &             *(FLOAT(IM1)*RHON1+FLN*(RHO**IM1)*(1.0D0
     &             -RHO**(N-2*IM1))))/(RHO**IM1)
      GO TO 456
  234  IF(IM1.NE.N-IM1)GO TO 345
      COVTC(I)=1.0D0-(FLN*(1.0D0-RHO*RHO)-2.0D0*RHO*RHON1)/
     &             (FLNS*RHOM1S*RHO**IM1)
      GO TO 456
  345  COVTC(I)=1.0D0-(FIRT+SECT-(THIT/FLOAT(N-IM1))
     &             *(FLOAT(IM1)*RHON1-FLN*RHO**(N-IM1)*(1.0D0
     &             -RHO**(2*IM1-N))))/(RHO**IM1)
  456  COVTC(I)=COVTC(I)*AK0(1)*RHO**IM1
  567   TSTATC(I)=DABS(COVT(I)-COVTC(I))/DSQRT(COVTV(I)/FISIM)
        BIASCO(I)=COVT(I)-AK0(IADD(I,1))
        BIASCT(I)=COVTC(I)-AK0(IADD(I,1))
        DIFBC(I)=BIASCO(I)-BIASCT(I)
        SEC(I)=DSQRT(COVTV(I)/FISIM)
        DBSEC(I)=2.0D0*SEC(I)-DABS(DIFBC)
  140   CONTINUE
C
        CTT(1)=VT
C
        CALL EVAINT(COVTC,AINTE,N)
        TSTIN=DABS(AINTT-AINTE)/DSQRT(AINTV/FISIM)
C
  141   WRITE(6,1100)ISIM
 1100   FORMAT(///T20,'FINAL RESULTS WITH',I5,2X,'SIMULATIONS')
        WRITE(6,1000)AMEANT,VART,STDT,BIASMO,SEM,DBSEM,
     &                   BIASVO,BIASVT,SEV,DBSEV
 1000   FORMAT(/T10,'ESTIMATED MEAN OF VARIATES GENERATED = ',
     &          D15.6/,T10,
     &          'AVERAGE ESTIMATED VARIANCE OF VARIATES GENERATED = ',
     &          D15.6/T10,'ESTIMATED STD. DEV. OF VARIATES',
     &          ' GENERATED = ',D15.6/T10,
     &          'OBSERVD BIAS FOR MEAN = ',D15.6/T10,
     &          'STANDARD ERROR FOR MEAN = ',D15.6/T10,
     &          'DIFF. IN BIAS DIFF. & 2 S.E. FOR MEAN = ',D15.6/T10,
     &          'OBSERVED BIAS FOR VAR. = ',D15.6/T10,
     &          'THEORETICAL BIAS FOR VAR. = ',D15.6/T10,
     &          'STANDARD ERROR FOR VAR. = ',D15.6/T10,
     &          'DIFF. IN BIAS DIFF. & 2 S.E. FOR VAR. = ',D15.6)
C
        IF(DABS(AMEAN).LE.0.0D0)GO TO 143
        DIM=((AMEAN-AMT)/AMEAN)*100.0D0
        DIFFM=((AMEAN-AMEANT)/AMEAN)*100.0D0
        GO TO 145
  143   DIM=-AMT*100.0D0
        DIFFM=-AMEANT*100.0D0
  145   WRITE(6,1010)VARTV,VARTC
 1010   FORMAT(/T10,'ESTIMATED VARIANCE OF VARIANCE = ',D15.6,/
     &          T10,'EXPECTED VALUE OF VARIANCE = ',D15.6)
        DIS=((STD-ST)/STD)*100.D0
        IF(AINTSC.EQ.0.0D0)GO TO 170
        DO 150 I=1,N
        L=IADD(I,1)
        DT(I)=((AK0(L)-CTT(I))/AK0(L))*100.0D0
  150   CONTINUE
        WRITE(6,1020)
 1020   FORMAT(/T30,'ESTIMATED COVARIANCES',/
     &          T2,'LAG',T7,'PAIRS',T16,'EST. COV.',
     &          T30,'ASS. COV.',T44,'EXP. COV.',T56,'VAR. OF COV.',
     &          T72,'OBS. BIAS',T85,'THEO. BIAS',
     &          T98,'S.E. OF MEAN',T117,'DIFF.')
        DO 160 K=1,N
        KM1=K-1
        WRITE(6,1030)KM1,ICOUNT(K),COVT(K),AK0(IADD(K,1)),COVTC(K),
     &                 COVTV(K),BIASCO(K),BIASCT(K),SEC(K),DBSEC(K)
  160   CONTINUE
 1030   FORMAT(T2,I3,T8,I3,T12,8D14.6)
        CALL EVAINT(COVT,AINTC,N)
        WRITE(6,1040)AINTT,AINTV
 1040   FORMAT(/T5,'ESTIMATED MEAN OF INTEGRAL SCALE',
     &          ' (FRACTION)  = ',D15.6,
     &          /T5,'VARIANCE OF INTEGRAL SCALE = ',D15.6)
        WRITE(6,1050)AINTC
 1050   FORMAT(T5,'INTEGRAL SCALE (FRACTION) EVALUATED',
     &          ' FROM AVG. COV.  = ',D15.6)
```

```
      WRITE(6,1055)AINTE
 1055 FORMAT(T5,'INTEGRAL SCALE (FRACTION) EVALUATED',
     &          ' FROM EXPECTED COV.   = ',D15.6)
C
  170  WRITE(6,1060)
 1060 FORMAT(///T20,'OVERALL RESULTS')
      WRITE(6,1070)AMT,VT,ST
 1070 FORMAT(/T10,'ESTIMATED OVERALL-MEAN   = ',
     &          D15.6/,T10,
     &          'ESTIMATED OVERALL-VARIANCE = ',
     &          D15.6/T10,'ESTIMATED OVERALL-STD. DEV.',
     &          ' = ',D15.6)
C
      WRITE(6,1080)DIM,DIS
 1080 FORMAT(/T10,'DIFFERENCE BETWEEN ASSIGNED AND ',
     &          'ESTIMATED MEAN (%) = ',D15.6/,T10,
     &          'DIFFERENCE BETWEEN ASSIGNED AND ESTIMATED',
     &          ' STD. DEV. (%) = ',D15.6)
      IF(AINTSC.EQ.0.0D0)GO TO 190
      WRITE(6,4000)
 4000 FORMAT(/T30,'ESTIMATED COVARIANCES',/
     &          T10,'LAG NO.',T20,'NO. OF PAIRS',T40,'EST. COV.',
     &          T60,'ASS. COV.',T80,'DIFF. IN COV. (%)')
      DO 180 K=1,N
      KM1=K-1
      WRITE(6,4010)KM1,IC(K),CTT(K),AK0(IADD(K,1)),DT(K)
  180  CONTINUE
 4010 FORMAT(T9,I5,T22,I5,T36,D15.6,T57,D15.6,T78,D15.6)
      CALL EVAINT(CTT,AIT,N)
      WRITE(6,1090)AIT
 1090 FORMAT(/T5,'INTEGRAL SCALE (FRACTION) EVALUATED',
     &          ' FROM TOTAL COV.   = ',D15.6//)
C
  190  STOP
      END
C
C.........................................................FUNCTION
      FUNCTION IADD(I,J)
      IMPLICIT REAL*8(A-H,O-Z)
      IADD=J+I*(I-1)/2
      RETURN
      END
C
      FUNCTION COVFUN(VAR,AL,FRDELX,AINTSC)
      IMPLICIT REAL*8(A-H,O-Z)
      COVFUN=VAR*DEXP(-AL*FRDELX/AINTSC)
      RETURN
      END
C
C.........................................................TEST
      SUBROUTINE TEST(Y,AK0,AMEAN,STD,AINTSC,N,IP,IPP,AMEANG,
     &                  VARG,COVG,ICOUNT,AINTG,XX,XX2,CO,SU,VARTC)
      IMPLICIT REAL*8(A-H,O-Z)
      EXTERNAL IADD
      DIMENSION Y(1),ICOUNT(500),COVG(500),SUMX(500)
      DIMENSION DIFFC(500),AK0(1),CO(500),SU(500)
C
C     TEST FOR MEAN AND VARIANCE
C
      XX=0.0D0
      XX2=0.0D0
      DO 10 I=1,N
      YY=Y(I)
      XX=XX+YY
      XX2=XX2+YY*YY
```

```
   10 CONTINUE
      AMEANG=XX/FLOAT(N)
      VARG=(XX2-XX*XX/FLOAT(N))/FLOAT(N-1)
      STDG=DSQRT(VARG)
      IF(IP.NE.0)GO TO 15
      WRITE(6,1000)AMEANG,VARG,STDG
 1000 FORMAT(/T10,'ESTIMATED MEAN OF VARIATES GENERATED = ',
     &          D15.6/,T10,
     &          'ESTIMATED VARIANCE OF VARIATES GENERATED = ',
     &          D15.6/T10,'ESTIMATED STD. DEV. OF VARIATES',
     &          ' GENERATED = ',D15.6)
C
      IF(DABS(AMEAN).LE.0.0D0)GO TO 12
      DIFFM=((AMEAN-AMEANG)/AMEAN)*100.0D0
      GO TO 14
   12 DIFFM=-AMEANG*100.0D0
   14 DIFFV=((VARTC-VARG)/VARTC)*100.0D0
      WRITE(6,1010)DIFFM,DIFFV
 1010 FORMAT(/T10,'DIFFERENCE BETWEEN ASSIGNED AND ',
     &          'ESTIMATED MEAN (%) = ',D15.6/,T10,
     &          'DIFFERENCE BETWEEN EXPECTED AND ESTIMATED',
     &          ' VARIANCE (%) = ',D15.6)
C
C     CALCULATE COVARIANCE VALUES
C
   15 CONTINUE
      IF(AINTSC.EQ.0.0D0)GO TO 70
      DO 40 K=1,N
      ICOUNT(K)=0
      COVG(K)=0.0D0
      SUMX(K)=0.0D0
C
C     NEW POINTS
C
      DO 20 J=1,N
      YY=Y(J)
C
      IDIFF=N-J
      IF(IDIFF.LT.K-1)GO TO 30
      COVG(K)=COVG(K)+YY*Y(J+K-1)
      SUMX(K)=SUMX(K)+YY+Y(J+K-1)
      ICOUNT(K)=ICOUNT(K)+1
   20 CONTINUE
   30 CO(K)=COVG(K)
      SU(K)=SUMX(K)
      COVG(K)=(COVG(K)-AMEANG*SUMX(K)+FLOAT(ICOUNT(K))
    &           *AMEANG*AMEANG)/FLOAT(ICOUNT(K))
   40 CONTINUE
      COVG(1)=COVG(1)*FLOAT(ICOUNT(1))/FLOAT(N-1)
      DO 45 I=1,N
      L=IADD(I,1)
      DIFFC(I)=((AK0(L)-COVG(I))/AK0(L))*100.0D0
   45 CONTINUE
C
      IF(IPP.NE.0)GO TO 60
      WRITE(6,1020)
 1020 FORMAT(/T30,'ESTIMATED COVARIANCES',/
     &          T10,'LAG NO.',T20,'NO. OF PAIRS',T40,'EST. COV.',
     &          T60,'ASS. COV.',T80,'DIFF. IN COV. (%)')
      DO 50 K=1,N
      WRITE(6,1030)K,ICOUNT(K),COVG(K),AK0(IADD(K,1)),DIFFC(K)
   50 CONTINUE
 1030 FORMAT(T9,I5,T22,I5,T36,D15.6,T57,D15.6,T78,D15.6)
   60 CALL EVAINT(COVG,AINTG,N)
      WRITE(6,1040)AINTG
 1040   FORMAT(/T5,'INTEGRAL SCALE (FRACTION)  = ',D15.6)
C
   70   RETURN
        END
C
C..........................................................UDUB
        SUBROUTINE UDUB(A,N,NBW,IER)
C       PURPOSE: FACTORS THE SYMMETRIC POSITIVE DEFINITE MATRIX
C                A AS U'DU WHERE U IS THE UPPER TRIANGULAR
C                MATRIX BY THE FORWARD ELIMINATION STEP OF
C                GARSSIAN ELIMINATION.  D IS A DIAGONAL MATRIX
C                WHOSE ELEMENTS ARE THE RECIPROCALS OF THE
C                DIAGONAL ELEMENTS OF U.
C       PARAMETERS: A          MATRIX TO BE FACTORED
C                                (CONTAINS U ON OUTPUT)
C                     N          NUMBER OF ROWS/COLUMNS OF A
C                     NBW        BANDWIDTH OF A
C                     IER        1 IF NULL PIVOT ENCOUNTERED
C                     IER        0 IF NO NULL PIVOT ENCOUNTERED
C     EXTERNALS   IADD(I,J) ADDRESS OF (I,J)TH ELEMENT OF A FOR I>0.
C
        IMPLICIT REAL*8(A-H,O-Z)
        EXTERNAL IADD
        DIMENSION A(1)
        NM1=N-1
        IER=1
C
C       FOR EACH COLUMN ELIMINATE ELEMENTS BELOW THE DIAGONAL OF A
C
        DO 30 K=1,NM1
        KP1=K+1
C
C       CALCULATE PIVOT AND TEST FOR ZERO OR NEGATIVE VALUE
C
        PIV=A(IADD(K,K))
        IF(PIV.LT.1.0D-13)RETURN
        PIV=1.0D0/PIV
C
C       CALCULATE POSITION IN COLUMN K OF LAST NON ZERO ELEMENT
C
        IL=K+NBW-1
        IF(IL.GT.N)IL=N
        DO 20 I=KP1,IL
C
C       CALCULATE MULTIPLIERS
C
        AMIK=A(IADD(I,K))*PIV
        DO 10 J=I,IL
C
C       UPDATE THE ELEMENTS IN ROW J
C
        A(IADD(J,I))=A(IADD(J,I))-AMIK*A(IADD(J,K))
   10   CONTINUE
   20   CONTINUE
   30   CONTINUE
        IER=0
        RETURN
        END
C
C..........................................................EVAINT
        SUBROUTINE EVAINT(COV,AINTG,N)
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION COV(1)
C
        FRDELX=1.0D0/FLOAT(N)
```

```
      AINTG=0.0D0
      DO 20 I=1,N
      IF(COV(I).LE.0.0D0)GO TO 30
      IF(COV(I+1).GT.COV(I))GO TO 30
      AINTG=AINTG+COV(I)
   20 CONTINUE
   30 AINTG=AINTG*FRDELX/COV(1)
      RETURN
      END
```

```
C**************   CF1-DCRF   **************
C***** PROGRAM TO SIMULATE TWO-PHASE FLOW IN A
C*****   COREFLOOD TEST (INCOMPRESSIBLE FLUIDS).
C
C       NAME: CF1-DCRF
C       BOUDARY CONDITION: CONSTANT RATE AND CONTANT PRESSURE AT
C                                THE OUTLET END.
C       INITIAL CONDITION: EQUAL PRESSURE IN ALL BLOCKS.
C
C       FOR CONSTANT PRESSURE AT BOTH ENDS, ONLY MINOR
C       MODIFICATION IS NEEDED.
C
C       FOR MONTE CARLO SIMULATION RUN.
C
        IMPLICIT REAL*8(A-H,O-Z)
        EXTERNAL PCFUN
        EXTERNAL AKOFUN
        EXTERNAL AKWFUN
        COMMON/INIT/PO(30,400),PW(30,400),PC(30,400),SO(30,400),
     &                 SW(30,400),POI
        COMMON/NLEV/PON(30),PWN(30),PCN(30),SON(30),
     &                 SWN(30)
        COMMON/FLPRO/DENO,DENW,VISO,VISW,GRAV
        COMMON/TAB/SAT(25),AKROT(25),AKRWT(25),PCOWT(25),MSAT
        COMMON/SIZE/DELX(30),DELXC,DELY,DELZ,DEPTH(30)
        COMMON/PORPER/POR(30),PERMX(30),TX(29)
        COMMON/COEFF/AD(30),AE(30),AF(30),B(30)
        COMMON/RATE/OIR,WIR,OPR,WPR,COI,CWI,COP,CWP
        COMMON/MOBIL/OD(30),OF(30),WD(30),WF(30),GOWT(30),
     &                  GWWT(30)
        COMMON/SWSO/SWIR(30),SOR(30)
        COMMON/CONST/AC,RF
        DIMENSION IHEDIN(80),VP(30)
        DIMENSION AMPO(400),VARPO(400),STDPO(400)
        DIMENSION AMPW(400),VARPW(400),STDPW(400)
        DIMENSION AMSW(400),VARSW(400),STDSW(400)
        DATA FT,FTMAX,ETI/0.D0,0.D0,0.D0/
        DATA IICODE,IOCODE/5,6/
        DATA IFILE1,IFILE2/7,8/
C
        READ(IICODE,4000)(IHEDIN(IH),IH=1,40)
        READ(IICODE,4020)ISIM,IHOMO,IOUT
        WRITE(IOCODE,4000)(IHEDIN(IH),IH=1,40)
        WRITE(IOCODE,4020)ISIM,IHOMO,IOUT
        DO 1000 K=1,ISIM
        ITT=0
        FT=0.0D0
        FTMAX=0.0D0
        ETI=0.0D0
        COP=0.0D0
        CWP=0.0D0
        COI=0.0D0
        CWI=0.0D0
        ETO=0.0D0
        CUMPO=0.0D0
        CUMPW=0.0D0
        CUMIO=0.0D0
        CUMIW=0.0D0
        AMBEO=0.0D0
        AMBEW=0.0D0
C
        DO 10 I=1,60
        AD(I)=0.0D0
        AE(I)=0.0D0
        AF(I)=0.0D0
```

```
      B(I)=0.0D0
      OD(I)=0.0D0
      OF(I)=0.0D0
      WD(I)=0.0D0
      WF(I)=0.0D0
      GOWT(I)=0.0D0
      GWWT(I)=0.0D0
   10 CONTINUE
C
      IF(K.NE.1)GO TO 16
      WRITE(IOCODE,5000)
      WRITE(IOCODE,5010)
      WRITE(IOCODE,5020)
 5000 FORMAT(//T30,69('*'),/T30,'*',T98,'*',/T30,'*',T98,'*')
 5010 FORMAT(T30,'*',T50,'COREFLOOD-TEST SIMULATION',T98,'*',
     &          /T30,'*',T47,'INCOMPRESSIBLE OIL-WATER SYSTEMS',
     &          T98,'*')
 5020 FORMAT(T30,'*',T98,'*',/T30,'*',T98,'*',/T30,69('*'),///)
C
C
C***** CORE AND BLOCK DIMENSIONS
C
      CALL GRID(IICODE,IOCODE,II)
C
C***** FLUID PROPERTIES
C
      READ(IICODE,4000)(IHEDIN(IH),IH=1,40)
 4000 FORMAT(40A2)
      READ(IICODE,4010)DENO,DENW,VISO,VISW,GRAV
 4010 FORMAT(5E15.6)
      READ(IICODE,4000)(IHEDIN(IH),IH=1,40)
C
C     FW    = INJECTED WATER RATE TO TOTAL INJECTED RATE
C     QIT   = TOTAL INJECTED RATE
C     POL   = SPECIFIED PRESSURE AT THE OUTLET END
C
      READ(IICODE,4015)FW,QIT,POL
      READ(IICODE,4000)(IHEDIN(IH),IH=1,40)
      READ(IICODE,4015)AC,RF
      WIR=QIT*FW
      OIR=QIT-WIR
C
C     IF RUN FOR STEADY STATE FLOW USE ISS=0, IF NOT ISS=1 .
C
      READ(IICODE,4000)(IHEDIN(IH),IH=1,40)
      READ(IICODE,4020)ISS
      IF(ISS.EQ.0)WRITE(IOCODE,4012)
 4012 FORMAT(/T10,'*** STEADY STATE FLOW ***'/)
 4015 FORMAT(3E15.6)
      WRITE(IOCODE,5030)
 5030 FORMAT(//T15,'DENO',T30,'DENW',T45,'VISO',T60,'VISW',
     &          T75,'GRAV. CONST.')
      WRITE(IOCODE,5040)DENO,DENW,VISO,VISW,GRAV
 5040 FORMAT(T12,E15.6,T27,E15.6,T42,E15.6,T57,E15.6,T77,E15.6)
C
C     IF FUNCTION FOR REL. PERM. AND CAP. PRESS. ARE GIVEN,
C     IFUNC=0, IF NOT IFUNC=1.
C
      READ(IICODE,4000)(IHEDIN(IH),IH=1,40)
      READ(IICODE,4020)IFUNC
 4020 FORMAT(3I5)
      IF(IFUNC.EQ.0)GO TO 15
C
C***** TABLE FOR RELATIVE PERMEABILITY AND CAPILLARY PRESSURE
C
```

```
      CALL TABLE(IICODE,IOCODE)
      GO TO 17
C
   15 WRITE(IOCODE,5045)
 5045 FORMAT(//T15,'USING FUNCTIONAL FORMS FOR',
     &          ' CAPILLARY PRESSURE AND RELATIVE',
     &          ' PERMEABILITIES',/)
   17 WRITE(IOCODE,5049)AC,RF
 5049 FORMAT(/T5,'CONSTANT FOR CAPILLARY FUNCTION (AC) = ',
     &          E15.6/T5,'RECOVERY FACTOR = ',E15.6)
      WRITE(IOCODE,5042)FW,QIT,POL
 5042 FORMAT(T5,'FW = ',E15.6/
     &          T5,'TOTAL INJECTION RATE = ',E15.6/
     &          T5,'OIL OUTLET PRESSURE = ',E15.6)
C
C***** INITIAL CONDITIONS
C
      READ(IICODE,4000)(IHEDIN(IH),IH=1,40)
      READ(IICODE,4025)POI
 4025 FORMAT(E15.6)
C
C***** TIME STEP CONTROL
C
      CALL TIME(IICODE,IOCODE,NN,FACT1,FACT2,TMAX,
     &              DSMAX,DPMAX)
C
C***** POROSITY AND PERMEABILITY
C
   16 CALL PARAM(IICODE,IOCODE,IFILE1,IFILE2,II,K,ISIM,IHOMO,IOUT)
C
      CALL SWSOIR(II)
C
C
C***** CALCULATE INTERBLOCK TRANSMISSIBILITIES
C
      CALL TRAN(II)
C
      DO 18 I=1,II
      SWI=SWIR(I)
      IF(IFUNC.EQ.0)GO TO 12
      CALL INTERP(SAT,PCOWT,MSAT,SWI,PCI)
      GO TO 14
   12 PCI=PCFUN(SWI,SWIR(I),SOR(I))
   14 PWI=POI-PCI
      PO(I,K)=POI
      PW(I,K)=PWI
      PC(I,K)=PCI
      SW(I,K)=SWI
      SO(I,K)=1.0D0-SWI
      PON(I)=POI
      PWN(I)=PWI
      PCN(I)=PCI
      SWN(I)=SWI
      SON(I)=1.0D0-SWI
   18 CONTINUE
      PO(II,K)=POL
C
      IF(K.NE.1)GO TO 19
      READ(IICODE,4000)(IHEDIN(IH),IH=1,40)
   19 NMAX=NN+1
C
      DO 500 N=1,NMAX
      NLOOP=N
C
C***** RECURRENT DATA
```

```
C
      IF(FT.LT.FTMAX)GO TO 20
      READ(IICODE,4030,END=1000)ICHANG,IWLREP,ISUMRY,IPOMAP,
     &       IPWMAP,IPCMAP,ISOMAP,ISWMAP
 4030 FORMAT(10I5)
      READ(IICODE,4015)DELT,DTMIN,DTMAX
      FTMAX=ETI+FLOAT(ICHANG)*DELT
   20 IF(N.EQ.1)DELTO=DELT
      IF(N.EQ.1.OR.ITT.EQ.1)GO TO 40
   30 IF(DSMC.LT.DSMAX.AND.DPMC.LT.DPMAX)DELT=DELT*FACT1
      IF(DSMC.GT.DSMAX.OR.DPMC.GT.DPMAX)DELT=DELT*FACT2
      IF(DELT.LT.DTMIN)DELT=DTMIN
      IF(DELT.GT.DTMAX)DELT=DTMAX
      IF(ETI+DELT.GT.FTMAX)DELT=FTMAX-ETI
   40 FT=ETI+DELT
      IF(ETI+DELT*0.5D0.GE.TMAX)GO TO 990
C
C***** ITFLAG COUNTS THE NUMBER OF TIME STEP REPITITIONS
C
      ITT=0
      ITFLAG=0
C***** REENTRY POINT FOR REPEATED TIME STEP
   50 CONTINUE
C
      DIV1=1.0D0/DELT
      IF(N.GT.1.OR.ITFLAG.GT.0)GO TO 70
      IF(ITT.EQ.1)GO TO 70
      RESVOL=0.0D0
      SCMO=0.0D0
      SCMW=0.0D0
      OOIP=0.0D0
      OWIP=0.0D0
      DO 60 I=1,II
      VP(I)=POR(I)*DELX(I)*DELY*DELZ
      RESVOL=RESVOL+VP(I)
      SCMO=SCMO+VP(I)*SO(I,K)
      SCMW=SCMW+VP(I)*SW(I,K)
      OOIP=OOIP+SON(I)*VP(I)
      OWIP=OWIP+SWN(I)*VP(I)
   60 CONTINUE
      IF(IOUT.EQ.0)GO TO 70
      WRITE(IOCODE,5050)OOIP,OWIP
 5050 FORMAT(/5X,'INITIAL FLUID VOLUMES: ',
     &          /10X,'OIL IN PLACE (m**3)',T60,E15.6,
     &          /10X,'WATER IN PLACE (m**3)',T60,E15.6/)
C
C***** PRINT OUT INITIAL ARRAY
C
   70 IF(N.EQ.1.AND.ITFLAG.LE.0)
     & CALL PRTPS(IOCODE,NLOOP,II,SCMOI,SCMWI,
     &                 AMBEO,AMBEW,DELTO,OP,WP,OI,WI,
     &                 SCMO,SCMW,ETI,CWORP,WORP,IPOMAP,IPWMAP,IPCMAP,
     &                 ISOMAP,ISWMAP,QPT,CQPT,DIFRAT,DIFCUM,K)
C
   80 CONTINUE
      IF(N.EQ.NMAX)GO TO 650
C
C***** CALCULATE COEFFICIENT MATRIX
C
      CALL COMAT(II,IFUNC,K)
C
C***** CALCULATE NEW OIL PRESSURE
C
      CALL BANDIN(II,QPT,K)
C
```

```
C***** CALCULATE OIL PRODUCTION RATE
C
      IIM1=II-1
C
C***** CALCULATE NEW FLUID SATURATIONS
C
   87 SCMO1=0.0D0
      SCMW1=0.0D0
      RESVOL=0.0D0
      DO 120 I=1,II
      PP=PO(I,K)
      VPP=VP(I)
      RESVOL=RESVOL+VPP
C
      DP1=0.0D0
      DP2=0.0D0
      IF((I-1).GT.0)DP1=PO(I-1,K)-PP
      IF((I+1).LE.II)DP2=PP-PO(I+1,K)
C
      DAODP=OD(I)*DP1-OF(I)*DP2
      DAWDP=WD(I)*DP1-WF(I)*DP2
      DTVP=DELT/VPP
C
      IF(FW.NE.1.0D0)GO TO 122
      SODIF=SON(I)-SOR(I)
      IF(DABS(SODIF).LT.0.0000001D0)GO TO 114
      IF(I.NE.1)GO TO 90
   88 SO(I,K)=SON(I)+DTVP*(OIR+DAODP+GOWT(I))
      GO TO 110
   90 IF(I.EQ.II)GO TO 100
      SO(I,K)=SON(I)+DTVP*(DAODP+GOWT(I))
      GO TO 110
  100 SO(I,K)=SO(I-1,K)
  110 SOD=SO(I,K)-SOR(I)
      IF(SOD.LT.-0.0000001D0)GO TO 113
  112 IF(SOD.LE.0.0D0)GO TO 114
      GO TO 116
  113 V1=SON(I)-SOR(I)
      V2=SON(I)-SO(I,K)
      DELTC=(V1/V2)*DELT
      IF(DELTC.GT.0.0D0)GO TO 154
      IF(DELTC.LT.0.0D0)GO TO 680
  114 SO(I,K)=SOR(I)
  116 SW(I,K)=1.0D0-SO(I,K)
      GO TO 128
C
  122 IF(I.NE.1)GO TO 124
      SO(I,K)=SON(I)+DTVP*(OIR+DAODP+GOWT(I))
      IF(SO(I,K).LE.SOR(I))SO(I,K)=SOR(I)
      GO TO 127
  124 IF(I.EQ.II)GO TO 125
      SO(I,K)=SON(I)+DTVP*(DAODP+GOWT(I))
      GO TO 127
  125 SO(I,K)=SO(I-1,K)
  127 SW(I,K)=1.0D0-SO(I,K)
C
  128 SSW=SW(I,K)
      SCMO1=SCMO1+VPP*SO(I,K)
      SCMW1=SCMW1+VPP*SW(I,K)
      IF(IFUNC.EQ.0)GO TO 115
      CALL INTERP(SAT,PCOWT,MSAT,SSW,PCOWI)
      GO TO 117
  115 PCOWI=PCFUN(SSW,SWIR(I),SOR(I))
  117 PC(I,K)=PCOWI
      PW(I,K)=PO(I,K)-PCOWI
  120   CONTINUE
C
        OPR=-OD(II)*(PO(IIM1,K)-PO(II,K))-GOWT(II)
      &      -(SON(II)-SO(II,K))*VP(II)/DELT
        WPR=QPT-OPR
C
C***** AUTOMATIC TIME STEP CONTROL
C
C
  135   PPM=0.0D0
        SOM=0.0D0
        SWM=0.0D0
        DO 150 I=1,II
        DPO=PO(I,K)-PON(I)
        DSO=SO(I,K)-SON(I)
        DSW=SW(I,K)-SWN(I)
        IF(DABS(DPO).LE.DABS(PPM))GO TO 130
        PPM=DPO
  130   IF(DABS(DSO).LE.DABS(SOM))GO TO 140
        SOM=DSO
  140   IF(DABS(DSW).LE.DABS(SWM))GO TO 150
        SWM=DSW
  150   CONTINUE
        DPMC=DABS(PPM)
        DSMC=DABS(SOM)
        IF(DSMC.LT.DABS(SWM))DSMC=DABS(SWM)
C
C***** REPEAT TIME STEP?
C
  152   IF(DSMC.LT.DSMAX.AND.DPMC.LT.DPMAX)GO TO 170
        IF(DELT.LE.DTMIN.OR.FACT2.GT.1.0D0)GO TO 170
        ITFLAG=ITFLAG+1
        DELT=DELT*FACT2
        IF(DELT.LT.DTMIN)DELT=DTMIN
        GO TO 156
  154   DELT=DELTC
        ITT=1
  156   FT=ETI+DELT
        IF(FT.GT.FTMAX)DELT=FTMAX-ETI
C
C***** RESET VARIABLES
C
        DO 160 I=1,II
        PO(I,K)=PON(I)
        PW(I,K)=PWN(I)
        PC(I,K)=PCN(I)
        SO(I,K)=SON(I)
        SW(I,K)=SWN(I)
  160   CONTINUE
        GO TO 50
C
C***** UPDATE OLD FLUID VOLUMES FOR MATERIAL BALANCE
C
  170   SCMOI=SCMO
        SCMWI=SCMW
C
C***** UPDATE NEW FLUID VOLUMES
C
        SCMO=SCMO1
        SCMW=SCMW1
C
        TOR=OIR+OPR
        TWR=WIR+WPR
        TOC=COI+COP
        TWC=CWI+CWP
```

```
C
C***** CALCULATE MATERIAL BALANCE ERRORS
C
        DELTO=DELT
        ETI=ETI+DELT
        CALL MATBAL(II,SCMO,SCMOI,SCMW,SCMWI,AMBEO,AMBEW,
     &                  DELTO,RESVOL,OP,WP,OI,WI,NLOOP,CWORP,WORP,
     &                  CWORI,WORI,QPT,CQPT,DIFRAT,DIFCUM)
C
C***** RATE REPORT
C
        IF(IWLREP.EQ.0)GO TO 175
        WRITE(IOCODE,5060)FT
  5060  FORMAT(/T5,10('*'),'RATE REPORT',10X,'ELAPSED TIME =',
     &          E15.6,2X,'SECOND FROM BEGINNING OF SIMULATION'/)
        WRITE(IOCODE,5070)
  5070  FORMAT(/T56,'------ RATE ------',22X,'------ ',
     &          'CUMULATIVE ------',/T5,'POSITION',T23,
     &          'PRESSURE',T41,'OIL',T59,'WATER',T77,'WOR',
     &          T95,'OIL',T113,'WATER',/T39,'m**3/s',T58,
     &          'm**3/s',T93,'m**3/s',T112,'m**3/s')
        WRITE(IOCODE,5080)PW(1,K),OIR,WIR,WORI,COI,CWI
  5080  FORMAT(T6,'INLET',T20,E15.6,T35,E15.6,T54,E15.6,T71,E15.6,
     &          T89,E15.6,T108,E15.6)
        WRITE(IOCODE,5090)PO(II,K),OPR,WPR,WORP,COP,CWP
  5090  FORMAT(T6,'OUTLET',T20,E15.6,T35,E15.6,T54,E15.6,T71,E15.6,
     &          T89,E15.6,T108,E15.6)
        WRITE(IOCODE,5100)TOR,TWR,TOC,TWC
  5100  FORMAT(T6,117('-'),/T6,'TOTAL',T35,E15.6,T54,E15.6,
     &          T89,E15.6,T108,E15.6)
C
C***** SUMMARY REPORT
C
   175  IF(ISUMRY.EQ.0)GO TO 180
        NLP=N+1
        CALL PRTPS(IOCODE,NLP,II,SCMOI,SCMWI,
     &                  AMBEO,AMBEW,DELTO,OP,WP,OI,WI,
     &                  SCMO,SCMW,ETI,CWORP,WORP,IPOMAP,IPWMAP,IPCMAP,
     &                  ISOMAP,ISWMAP,QPT,CQPT,DIFRAT,DIFCUM,K)
   180  CONTINUE
C
C***** UPDATE ARRAYS
C
        DO 190 I=1,II
        PON(I)=PO(I,K)
        PWN(I)=PW(I,K)
        PCN(I)=PC(I,K)
        SON(I)=SO(I,K)
        SWN(I)=SW(I,K)
   190  CONTINUE
        QDIF=(WIR+WPR)/WIR
        IF(DABS(QDIF).LE.1.0D-05)GO TO 990
C
   500  CONTINUE
   900  IPOMAP=1
        IPWMAP=1
        IPCMAP=0
        ISOMAP=0
        ISWMAP=1
        CALL PRTPS(IOCODE,NLOOP,II,SCMOI,SCMWI,
     &                  AMBEO,AMBEW,DELTO,OP,WP,OI,WI,
     &                  SCMO,SCMW,ETI,CWORP,WORP,IPOMAP,
     &                  IPWMAP,IPCMAP,ISOMAP,ISWMAP,
     &                  QPT,CQPT,DIFRAT,DIFCUM,K)
   990  WRITE(IOCODE,3333)ETI,WIR,OIR,WPR,OPR
```

```
3333  FORMAT(2X,'TOTAL TIME =',D14.6,3X,'WIR =',D14.6,3X,
     &          'OIR =',D14.6,3X,'WPR =',D14.6,3X,'OPR =',D14.6)
1000  CONTINUE
      IF(K.EQ.1)GO TO 700
C
      WRITE(9,5105)
5105  FORMAT(/T20,'PRESSURE AND SATURATION RESULTS'//
     &          1X,'BLK',T9,'MEAN PO',T23,'VAR PO',
     &          T37,'STD PO',T51,'MEAN PW',T65,'VAR PW',
     &          T79,'STD PW',T93,'MEAN SW',T107,'VAR SW',
     &          T121,'STD SW'/)
      FLISIM=FLOAT(ISIM)
      FLISM1=FLOAT(ISIM-1)
      AMMSW=0.0D0
      DO 260 I=1,II
      AMPO(I)=0.0D0
      VARPO(I)=0.0D0
      AMPW(I)=0.0D0
      VARPW(I)=0.0D0
      AMSW(I)=0.0D0
      VARSW(I)=0.0D0
      DO 200 K=1,ISIM
      AMPO(I)=AMPO(I)+PO(I,K)
      VARPO(I)=VARPO(I)+PO(I,K)*PO(I,K)
      AMPW(I)=AMPW(I)+PW(I,K)
      VARPW(I)=VARPW(I)+PW(I,K)*PW(I,K)
      AMSW(I)=AMSW(I)+SW(I,K)
      VARSW(I)=VARSW(I)+SW(I,K)*SW(I,K)
 200  CONTINUE
      VARPO(I)=(VARPO(I)-AMPO(I)*AMPO(I)/FLISIM)/FLISM1
      AMPO(I)=AMPO(I)/FLISIM
      IF(VARPO(I).LE.0.0D0)GO TO 210
      STDPO(I)=DSQRT(VARPO(I))
      GO TO 215
 210  STDPO(I)=0.0D0
 215  VARPW(I)=(VARPW(I)-AMPW(I)*AMPW(I)/FLISIM)/FLISM1
      AMPW(I)=AMPW(I)/FLISIM
      IF(VARPW(I).LE.0.0D0)GO TO 217
      STDPW(I)=DSQRT(VARPW(I))
      GO TO 220
 217  STDPW(I)=0.0D0
 220  VARSW(I)=(VARSW(I)-AMSW(I)*AMSW(I)/FLISIM)/FLISM1
      AMSW(I)=AMSW(I)/FLISIM
      AMMSW=AMMSW+AMSW(I)
      IF(VARSW(I).LE.0.0D0)GO TO 230
      STDSW(I)=DSQRT(VARSW(I))
      GO TO 250
 230  STDSW(I)=0.0D0
 250  CONTINUE
      WRITE(9,5110)I,AMPO(I),VARPO(I),STDPO(I),
     &                        AMPW(I),VARPW(I),STDPW(I),
     &                        AMSW(I),VARSW(I),STDSW(I)
      WRITE(2,5115)(PO(I,K),K=1,ISIM)
      WRITE(3,5115)(PW(I,K),K=1,ISIM)
      WRITE(4,5115)(SW(I,K),K=1,ISIM)
 260  CONTINUE
      AMMSW=AMMSW/FLOAT(II)
      WRITE(9,5112)AMMSW
5112  FORMAT(/T79,'MEAN',T89,E14.6/)
5110  FORMAT(1X,I3,9(E14.6))
5115  FORMAT(6E13.6)
      GO TO 700
 600  WRITE(IOCODE,5120)
5120  FORMAT(//T5,'MAXIMUM TIME LIMIT IS EXCEEDED'//)
      GO TO 700
```

```
  650 WRITE(IOCODE,5130)
 5130 FORMAT(//T5,'MAXIMUM NUMBER OF TIME STEP IS EXCEEDED'//)
      GO TO 700
  670 WRITE(IOCODE,5140)OPRC,QPT
 5140 FORMAT(//T5,'OPRC = ',E16.6,20X,'QPT = ',E16.6/)
      GO TO 700
  680 WRITE(IOCODE,5150)
 5150 FORMAT(//T5,'DELTC IS NEGATIVE')
      GO TO 700
  690 WRITE(IOCODE,5160)
 5160 FORMAT(//T5,'WATER INJECTION RATE IS TOO HIGH'/)
      GO TO 700
  695 WRITE(IOCODE,5170)
 5170 FORMAT(//T5,'OIL PRODECTION RATE IS TO LOW'/)
  700 STOP
      END
C
C........................................................BANDIN
C
      SUBROUTINE BANDIN(NX,QTP,K)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/COEFF/AD(30),AE(30),AF(30),B(30)
      COMMON/INIT/PO(30,400),PW(30,400),PC(30,400),SO(30,400),
     &                SW(30,400),POI
      DIMENSION PVEC(30)
C
      CALL THOMAS(AE,AF,AD,B,PVEC,NX)
      NXM=NX-1
      DO 50 I=1,NXM
      PO(I,K)=PVEC(I)
   50 CONTINUE
      QTP=PVEC(NX)
      RETURN
      END
C
C........................................................COMAT
C
      SUBROUTINE COMAT(II,IFUNC,K)
      IMPLICIT REAL*8(A-H,O-Z)
      EXTERNAL PCFUN
      EXTERNAL AKOFUN
      EXTERNAL AKWFUN
      COMMON/FLPRO/DENO,DENW,VISO,VISW,GRAV
      COMMON/NLEV/PON(30),PWN(30),PCN(30),SON(30),SWN(30)
      COMMON/SIZE/DELX(30),DELXC,DELY,DELZ,DEPTH(30)
      COMMON/PORPER/POR(30),PERMX(30),TX(29)
      COMMON/INIT/PO(30,400),PW(30,400),PC(30,400),SO(30,400),
     &                SW(30,400),POI
      COMMON/TAB/SAT(25),AKROT(25),AKRWT(25),PCOWT(25),MSAT
      COMMON/COEFF/AD(30),AE(30),AF(30),B(30)
      COMMON/RATE/OIR,WIR,OPR,WPR,COI,CWI,COP,CWP
      COMMON/MOBIL/OD(30),OF(30),WD(30),WF(30),GOWT(30),
     &                  GWWT(30)
      COMMON/SWSO/SWIR(30),SOR(30)
C
C***** COEFFICIENT MATRIX
C
C***** FOR BLOCK 1
C
      PP=PO(1,K)
      SSO=SO(1,K)
      SSW=SW(1,K)
      SWIRC=SWIR(1)
      SORC=SOR(1)
      IF(IFUNC.EQ.0)GO TO 10
      CALL  INTERP(SAT,PCOWT,MSAT,SSW,PCOW)
      GO  TO  20
   10 PCOW=PCFUN(SSW,SWIRC,SORC)
   20 P2=PO(2,K)
      SO2=SO(2,K)
      SW2=SW(2,K)
      SWIR2=SWIR(2)
      SOR2=SOR(2)
      IF(IFUNC.EQ.0)GO  TO  30
      CALL  INTERP(SAT,PCOWT,MSAT,SW2,PCOW2)
      GO  TO  40
   30 PCOW2=PCFUN(SW2,SWIR2,SOR2)
   40 FACT=-GRAV*(DEPTH(1)-DEPTH(2))
      GOW2=DENO*FACT
      GWW2=DENW*FACT-PCOW+PCOW2
C
      P22=PP-P2
      HO2=P22+GOW2
      HW2=P22+GWW2
C
      IF(IFUNC.EQ.0)GO  TO  50
      IF(HO2.GE.0.0D0)CALL  INTERP(SAT,AKROT,MSAT,SSO,AKRO2)
      IF(HO2.LT.0.0D0)CALL  INTERP(SAT,AKROT,MSAT,SO2,AKRO2)
      IF(HW2.GE.0.0D0)CALL  INTERP(SAT,AKRWT,MSAT,SSW,AKRW2)
      IF(HW2.LT.0.0D0)CALL  INTERP(SAT,AKRWT,MSAT,SW2,AKRW2)
      GO  TO  60
   50 IF(HO2.GE.0.0D0)AKRO2=AKOFUN(SSO,SWIRC,SORC)
      IF(HO2.LT.0.0D0)AKRO2=AKOFUN(SO2,SWIR2,SOR2)
      IF(HW2.GE.0.0D0)AKRW2=AKWFUN(SSW,SWIRC,SORC)
      IF(HW2.LT.0.0D0)AKRW2=AKWFUN(SW2,SWIR2,SOR2)
   60 AMO2=AKRO2/VISO
      AMW2=AKRW2/VISW
C
      AOF=TX(1)*AMO2
      AWF=TX(1)*AMW2
C
      AF(1)=AOF+AWF
      AE(1)=-(AOF+AWF)
C
      AO2=GOW2*AOF
      AW2=GWW2*AWF
      GOWT(1)=-AO2
      GWWT(1)=-AW2
C
      B(1)=-OIR-WIR+AO2+AW2
C
      OF(1)=AOF
      WF(1)=AWF
      OD(1)=0.0D0
      WD(1)=0.0D0
C
C***** FOR  BLOCK  2  TO  BLOCK  (II-1)
C
      IIM1=II-1
      DO  170  I=2,IIM1
      PP=PO(I,K)
      SSO=SO(I,K)
      SSW=SW(I,K)
      SWIRC=SWIR(I)
      SORC=SOR(I)
      IF(IFUNC.EQ.0)GO  TO  70
      CALL  INTERP(SAT,PCOWT,MSAT,SSW,PCOW)
      GO  TO  80
   70 PCOW=PCFUN(SSW,SWIRC,SORC)
   80 P1=PO(I-1,K)
```

```
      SO1=SO(I-1,K)
      SW1=SW(I-1,K)
      SWIR1=SWIR(I-1)
      SOR1=SOR(I-1)
      IF(IFUNC.EQ.0)GO TO 90
      CALL  INTERP(SAT,PCOWT,MSAT,SW1,PCOW1)
      GO TO 100
   90 PCOW1=PCFUN(SW1,SWIR1,SOR1)
C
  100  FACT=-GRAV*(DEPTH(I-1)-DEPTH(I))
      GOW1=DENO*FACT
      GWW1=DENW*FACT-PCOW1+PCOW
C
      P11=P1-PP
      HO1=P11+GOW1
      HW1=P11+GWW1
C
      IF(IFUNC.EQ.0)GO  TO  110
      IF(HO1.GE.0.0D0)CALL  INTERP(SAT,AKROT,MSAT,SO1,AKRO1)
      IF(HO1.LT.0.0D0)CALL  INTERP(SAT,AKROT,MSAT,SSO,AKRO1)
      IF(HW1.GE.0.0D0)CALL  INTERP(SAT,AKRWT,MSAT,SW1,AKRW1)
      IF(HW1.LT.0.0D0)CALL  INTERP(SAT,AKRWT,MSAT,SWO,AKRW1)
      GO  TO  120
  110  IF(HO1.GE.0.0D0)AKRO1=AKOFUN(SO1,SWIR1,SOR1)
      IF(HO1.LT.0.0D0)AKRO1=AKOFUN(SSO,SWIRC,SORC)
      IF(HW1.GE.0.0D0)AKRW1=AKWFUN(SW1,SWIR1,SOR1)
      IF(HW1.LT.0.0D0)AKRW1=AKWFUN(SWO,SWIRC,SORC)
  120  AMO1=AKRO1/VISO
      AMW1=AKRW1/VISW
C
      AOD=TX(I-1)*AMO1
      AWD=TX(I-1)*AMW1
C
      P2=PO(I+1,K)
      SO2=SO(I+1,K)
      SW2=SW(I+1,K)
      SWIR2=SWIR(I+1)
      SOR2=SOR(I+1)
      IF(IFUNC.EQ.0)GO  TO  130
      CALL  INTERP(SAT,PCOWT,MSAT,SW2,PCOW2)
      GO  TO  140
  130  PCOW2=PCFUN(SW2,SWIR2,SOR2)
C
  140  FACT=-GRAV*(DEPTH(I)-DEPTH(I+1))
      GOW2=DENO*FACT
      GWW2=DENW*FACT-PCOW+PCOW2
C
      P22=PP-P2
      HO2=P22+GOW2
      HW2=P22+GWW2
C
      IF(IFUNC.EQ.0)GO  TO  150
      IF(HO2.GE.0.0D0)CALL  INTERP(SAT,AKROT,MSAT,SSO,AKRO2)
      IF(HO2.LT.0.0D0)CALL  INTERP(SAT,AKROT,MSAT,SO2,AKRO2)
      IF(HW2.GE.0.0D0)CALL  INTERP(SAT,AKRWT,MSAT,SSW,AKRW2)
      IF(HW2.LT.0.0D0)CALL  INTERP(SAT,AKRWT,MSAT,SW2,AKRW2)
      GO  TO  160
  150  IF(HO2.GE.0.0D0)AKRO2=AKOFUN(SSO,SWIRC,SORC)
      IF(HO2.LT.0.0D0)AKRO2=AKOFUN(SO2,SWIR2,SOR2)
      IF(HW2.GE.0.0D0)AKRW2=AKWFUN(SSW,SWIRC,SORC)
      IF(HW2.LT.0.0D0)AKRW2=AKWFUN(SW2,SWIR2,SOR2)
  160  AMO2=AKRO2/VISO
      AMW2=AKRW2/VISW
C
      AOF=TX(I)*AMO2
      AWF=TX(I)*AMW2
C
      AD(I)=AOD+AWD
      AF(I)=AOF+AWF
      IF(I.EQ.IIM1)AF(I)=0.0D0
      AE(I)=-(AOD+AWD+AOF+AWF)
      AO1=GOW1*AOD
      AW1=GWW1*AWD
      AO2=GOW2*AOF
      AW2=GWW2*AWF
      GOWT(I)=AO1-AO2
      GWWT(I)=AW1-AW2
C
      B(I)=-AO1-AW1+AO2+AW2
      IF(I.EQ.IIM1)B(I)=B(I)-(AOF+AWF)*PO(II,K)
C
      OD(I)=AOD
      WD(I)=AWD
      OF(I)=AOF
      WF(I)=AWF
  170 CONTINUE
C
C*****  FOR  BLOCK  II
C
      PP=PO(II,K)
      SSO=SO(II,K)
      SSW=SW(II,K)
      SWIRC=SWIR(II)
      SORC=SOR(II)
      IF(IFUNC.EQ.0)GO TO 180
      CALL  INTERP(SAT,PCOWT,MSAT,SSW,PCOW)
      GO  TO  190
  180 PCOW=PCFUN(SSW,SWIRC,SORC)
  190 P1=PO(II-1,K)
      SO1=SO(II-1,K)
      SW1=SW(II-1,K)
      SWIR1=SWIR(II-1)
      SOR1=SOR(II-1)
      IF(IFUNC.EQ.0)GO TO  200
      CALL  INTERP(SAT,PCOWT,MSAT,SW1,PCOW1)
      GO  TO  210
  200 PCOW1=PCFUN(SW1,SWIR1,SOR1)
C
  210 FACT=-GRAV*(DEPTH(II-1)-DEPTH(II))
      GOW1=DENO*FACT
      GWW1=DENW*FACT-PCOW1+PCOW
C
      P11=P1-PP
      HO1=P11+GOW1
      HW1=P11+GWW1
C
      IF(IFUNC.EQ.0)GO  TO  220
      IF(HO1.GE.0.0D0)CALL  INTERP(SAT,AKROT,MSAT,SO1,AKRO1)
      IF(HO1.LT.0.0D0)CALL  INTERP(SAT,AKROT,MSAT,SSC,AKRO1)
      IF(HW1.GE.0.0D0)CALL  INTERP(SAT,AKRWT,MSAT,SW1,AKRW1)
      IF(HW1.LT.0.0D0)CALL  INTERP(SAT,AKRWT,MSAT,SSW,AKRW1)
      GO  TO  230
C
  220 IF(HO1.GE.0.0D0)AKRO1=AKOFUN(SO1,SWIR1,SOR1)
      IF(HO1.LT.0.0D0)AKRO1=AKOFUN(SSO,SWIRC,SORC)
      IF(HW1.GE.0.0D0)AKRW1=AKWFUN(SW1,SWIR1,SOR1)
      IF(HW1.LT.0.0D0)AKRW1=AKWFUN(SSW,SWIRC,SORC)
  230 AMO1=AKRO1/VISO
      AMW1=AKRW1/VISW
C
```

```
      AOD=TX(II-1)*AMO1
      AWD=TX(II-1)*AMW1
C
      AD(II)=AOD+AWD
      AE(II)=1.0D0
C
      AO1=GOW1*AOD
      AW1=GWW1*AWD
      GOWT(II)=AO1
      GWWT(II)=AW1
C
      B(II)=AD(II)*PO(II,K)-AO1-AW1
C
      OD(II)=AOD
      WD(II)=AWD
      OF(II)=0.0D0
      WF(II)=0.0D0
C
      RETURN
      END
C
C.......................................................FUNCTION
C
      FUNCTION  PCFUN(SW,SWIR,SOR)
      IMPLICIT  REAL*8(A-H,O-Z)
      COMMON/CONST/AC,RF
      B1=(1.0D0-SWIR-SOR)/(1.0D0-SWIR)
      RCPC=AC*DSQRT(B1)
      IF(SW.LE.SWIR)SW=SWIR
      S=(SW-SWIR)/(1.0D0-SWIR)+1.0D-03
      PCFUN=RCPC/DSQRT(S)-AC
      RETURN
      END
C
      FUNCTION  AKOFUN(SO,SWIR,SOR)
      IMPLICIT  REAL*8(A-H,O-Z)
      COMMON/CONST/AC,RF
      SIMB=(1.0D0-SO-SWIR)/(1.0D0-SWIR)
      RF1=1.0D0-RF
      IF(SIMB.GE.RF)SIMB=RF
      COMP=4.0D0*RF1*SIMB
      SDR=(1.0D0-DSQRT(1.0D0-COMP))/(2.0D0*RF1)
      T1=1.0D0-SDR
      AKOFUN=T1*T1*(1.0D0-SDR*SDR)
   20 IF(SO.LE.SOR)AKOFUN=0.0D0
      RETURN
      END
C
      FUNCTION  AKWFUN(SW,SWIR,SOR)
      IMPLICIT  REAL*8(A-H,O-Z)
      S=(SW-SWIR)/(1.0D0-SWIR)
      AKWFUN=S*S*S*S
      IF(SW.LE.SWIR)AKWFUN=0.0D0
      RETURN
      END
C
C...........................................................GRID
C
      SUBROUTINE  GRID(IICODE,IOCODE,II)
      IMPLICIT  REAL*8(A-H,O-Z)
      COMMON/SIZE/DELX(30),DELXC,DELY,DELZ,DEPTH(30)
      DIMENSION  IHEDIN(80)
      READ(IICODE,2000)(IHEDIN(IH),IH=1,40)
 2000 FORMAT(40A2)
      WRITE(IOCODE,3000)(IHEDIN(IH),IH=1,40)
 3000 FORMAT(T24,82('*'),/T24,'*',T105,'*',/T24,'*',40A2,T105,
     &          '*',/T24,82('*'),//)
C
C***** READ IN NUMBER OF BLOCKS AND SIZE
C
      READ(IICODE,2010)II
 2010 FORMAT(I5)
      READ(IICODE,2000)(IHEDIN(IH),IH=1,40)
      READ(IICODE,2020)DELXC,DELY,DELZ
 2020 FORMAT(3E15.6)
C
C***** ASSIGN VALUES FOR DELX
C
      IIM1=II-1
      IIM2=II-2
      DELX(1)=0.25D0*DELXC
      DELX(2)=0.75D0*DELXC
      DELX(IIM1)=0.75D0*DELXC
      DELX(II)=0.25D0*DELXC
      DO 10 I=3,IIM2
      DELX(I)=DELXC
   10 CONTINUE
      WRITE(IOCODE,3010)DELXC
 3010 FORMAT(T15,'GRID BLOCK LENGTH      (DX) IS INITIALLY SET AT',
     &          3X,E13.6,3X,'m',3X,'FOR ALL NODES'/)
      WRITE(IOCODE,3020)DELY
 3020 FORMAT(T15,'GRID BLOCK WIDTH       (DY) IS INITIALLY SET AT',
     &          3X,E13.6,3X,'m',3X,'FOR ALL NODES'/)
      WRITE(IOCODE,3030)DELZ
 3030 FORMAT(T15,'GRID BLOCK THICKNESS (DZ) IS INITIALLY SET AT',
     &          3X,E13.6,3X,'m',3X,'FOR ALL NODES'/)
C
C***** DEPTH
C
      READ(IICODE,2000)(IHEDIN(IH),IH=1,40)
      READ(IICODE,2010)KDEP
      IF(KDEP.NE.0)GO TO 30
      READ(IICODE,2030)DEPC
 2030 FORMAT(E15.6)
      DO 20 I=1,II
      DEPTH(I)=DEPC
   20 CONTINUE
      GO TO 40
   30 READ(IICODE,2040)(DEPTH(I),I=1,II)
 2040 FORMAT(5E15.6)
   40 WRITE(IOCODE,3040)
 3040 FORMAT(//T15,'***********    DEPTH TO MID POINTS    ***********'/)
      WRITE(IOCODE,3050)(DEPTH(I),I=1,II)
 3050 FORMAT(8E15.6)
      RETURN
      END
C
C...........................................................INTERP
C
      SUBROUTINE INTERP(X,Y,N,XO,YO)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(25),Y(25)
      IF(XO.GE.X(N))YO=Y(N)
      IF(XO.GE.X(N))RETURN
      DO 10 I=2,N
      IF(XO.GE.X(I))GO TO 10
      YO=Y(I-1)+(XO-X(I-1))*(Y(I)-Y(I-1))/(X(I)-X(I-1))
      RETURN
   10 CONTINUE
      END
```

```
C
C.............................................................MATBAL
C
      SUBROUTINE MATBAL(II,SCMO,SCMOI,SCMW,SCMWI,AMBEO,AMBEW,
     &                       DELTO,RESVOL,OP,WP,OI,WI,NLOOP,CWORP,
     &                       WORP,CWORI,WORI,QPT,CQPT,DIFRAT,DIFCUM)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/NLEV/PON(30),PWN(30),PCN(30),SON(30),SWN(30)
      COMMON/INIT/PO(30,400),PW(30,400),PC(30,400),SO(30,400),
     &              SW(30,400),POI
      COMMON/RATE/OIR,WIR,OPR,WPR,COI,CWI,COP,CWP
      COMMON/MOBIL/OD(30),OF(30),WD(30),WF(30),GOWT(30),
     &              GWWT(30)
C
      OI=OIR*DELTO
      WI=WIR*DELTO
      OP=OPR*DELTO
      WP=WPR*DELTO
      COI=COI+OI
      CWI=CWI+WI
      COP=COP+OP
      CWP=CWP+WP
C
      DENOM1=SCMOI+OP+OI
      IF(DENOM1.LT.1.D-22)GO TO 10
      AMBEO=(SCMO/DENOM1-1.0D0)*100.0D0
   10 CONTINUE
      DENOM2=SCMWI+WP+WI
      IF(DENOM2.LT.1.D-22)GO TO 20
      AMBEW=(SCMW/DENOM2-1.0D0)*100.0D0
   20 CONTINUE
      IF(OP.EQ.0.0D0)WORP=0.0D0
      IF(OP.EQ.0.0D0)GO TO 30
      WORP=WP/OP
   30 IF(COP.EQ.0.0D0)CWORP=0.0D0
      IF(COP.EQ.0.0D0)GO TO 40
      CWORP=CWP/COP
   40 IF(OI.EQ.0.0D0)WORI=0.0D0
      IF(OI.EQ.0.0D0)GO TO 50
      WORI=WI/OI
   50 IF(COI.EQ.0.0D0)CWORI=0.0D0
      IF(COI.EQ.0.0D0)GO TO 60
      CWORI=CWI/COI
   60 CONTINUE
C
      CQPT=QPT*DELTO
      DIFRAT=QPT-OPR-WPR
      DIFCUM=CQPT-OP-WP
C
      RETURN
      END
C
C..............................................................PARAM
C
      SUBROUTINE PARAM(IICODE,IOCODE,IFILE1,IFILE2,II,K,ISIM,IHOMO,
     &                       IOUT)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/PORPER/POR(30),PERMX(30),TX(29)
      DIMENSION IHEDIN(80)
C
      IIM1=II-1
      IIM2=II-2
      IF(IHOMO.NE.0)GO TO 5
      READ(IICODE,2000)(IHEDIN(I),I=1,40)
 2000 FORMAT(40A2)
```

```
C
C***** READ IN CODES FOR POROSITY AND PERMEABILITY
C
        READ(IICODE,2010)KPOR,KPERMX
 2010   FORMAT(2I5)
C
C***** POROSITY (POR)
C
        GO TO 7
     5   KPOR=1
        KPERMX=1
     7   IF(KPOR.NE.0)GO TO 20
        READ(IICODE,2020)PORC
 2020   FORMAT(E15.6)
        DO 10 I=1,II
        POR(I)=PORC
   10   CONTINUE
        WRITE(IOCODE,3000)
 3000   FORMAT(/)
        WRITE(IOCODE,3010)PORC
 3010   FORMAT(//T15,'POROSITY (POR) IS INITIALLY SET AT',E15.6,
     &          'FOR ALL NODES'/)
        GO TO 30
   20   READ(IFILE1,2030)(POR(I),I=2,IIM1)
        POR(1)=POR(2)
        POR(II)=POR(IIM1)
 2030   FORMAT(5E15.6)
        IF(IOUT.EQ.0)GO TO 30
        WRITE(IOCODE,3020)
 3020   FORMAT(/T15,'***********     POROSITY DISTRIBUTION    ***',
     &          '********')
        WRITE(IOCODE,3030)(POR(I),I=1,II)
 3030   FORMAT(8E15.6)
   30   CONTINUE
C
C***** PERMEABILITY (PERMX)
C
        IF(KPERMX.NE.0)GO TO 50
        READ(IICODE,2020)PERMXC
        DO 40 I=1,II
        PERMX(I)=PERMXC
   40   CONTINUE
        WRITE(IOCODE,3040)PERMXC
 3040   FORMAT(//T15,'PERMEABILITY (PERMX) IS INITIALLY SET AT',
     &          E15.6,'FOR ALL NODES'/)
        GO TO 60
   50   READ(IFILE2,2030)(PERMX(I),I=2,IIM1)
        PERMX(1)=PERMX(2)
        PERMX(II)=PERMX(IIM1)
        IF(IOUT.EQ.0)GO TO 60
        WRITE(IOCODE,3050)
 3050   FORMAT(/T15,'***********     PERMEABILITY (PERMX) DISTRIBUTION'
     &          ,'     ***********')
        WRITE(IOCODE,3030)(PERMX(I),I=1,II)
   60   RETURN
        END
C
C.........................................................PRTPS
C
        SUBROUTINE PRTPS(IOCODE,NLOOP,II,SCMOI,SCMWI,
     &                       AMBEO,AMBEW,DELTO,OP,WP,OI,WI,
     &                       SCMO,SCMW,ETI,CWORP,WORP,IPOMAP,
     &                       IPWMAP,IPCMAP,ISOMAP,ISWMAP,QPT,CQPT,
     &                       DIFRAT,DIFCUM,K)
        IMPLICIT REAL*8(A-H,O-Z)
```

```
      COMMON/INIT/PO(30,400),PW(30,400),PC(30,400),SO(30,400),
     &                  SW(30,400),POI
      COMMON/NLEV/PON(30),PWN(30),PCN(30),SON(30),SWN(30)
      COMMON/RATE/OIR,WIR,OPR,WPR,COI,CWI,COP,CWP
C
      POM=0.0D0
      PWM=0.0D0
      PCM=0.0D0
      SOM=0.0D0
      SWM=0.0D0
      IPOM=1
      IPWM=1
      IPCM=1
      ISOM=1
      ISWM=1
C
      IF(NLOOP.EQ.1)GO TO 60
      DO 50 I=1,II
      DPO=PO(I,K)-PON(I)
      DPW=PW(I,K)-PWN(I)
      DPC=PC(I,K)-PCN(I)
      DSO=SO(I,K)-SON(I)
      DSW=SW(I,K)-SWN(I)
      IF(DABS(DPO).LE.DABS(POM))GO TO 10
      POM=DPO
      IPOM=I
   10 IF(DABS(DPW).LE.DABS(PWM))GO TO 20
      PWM=DPW
      IPWM=I
   20 IF(DABS(DPC).LE.DABS(PCM))GO TO 30
      PCM=DPC
      IPCM=I
   30 IF(DABS(DSO).LE.DABS(SOM))GO TO 40
      SOM=DSO
      ISOM=I
   40 IF(DABS(DSW).LE.DABS(SWM))GO TO 50
      SWM=DSW
      ISWM=I
   50 CONTINUE
C
      WRITE(IOCODE,3000)
 3000 FORMAT(//T30,69('*'),/T30,'*',T98,'*',/T30,'*',T98,'*')
      WRITE(IOCODE,3010)
 3010 FORMAT(T30,'*',T45,'SUMMARY REPORT: 1-D COREFLOOD',
     &           T98,'*')
      WRITE(IOCODE,3020)
 3020 FORMAT(T30,'*',T98,'*',/T30,'*',T98,'*',T30,69('*')///)
C
   55 NLM=NLOOP-1
      WRITE(IOCODE,3030)ETI,NLM,DELTO,IPOM,POM,IPWM,PWM,IPCM,PCM,
     &                            ISOM,SOM,ISWM,SWM,AMBEO,AMBEW
 3030 FORMAT(/T2,'ELAPSED TIME (SEC)',T28,'=',E13.6,T43,
     &                 'TIME STEP NUMBER',T69,'=',I5,T84,
     &                 'TIME STEP SIZE (SEC)',T110,'=',E13.6,
     &         //T2,'OIL DPMAX (',I3,')',T28,'=',E13.6,T43,
     &                 'WATER DPMAX (',I3,')',T69,'=',E13.6,T84,
     &                 'CAP. DPCMAX (',I3,')',T110,'=',E13.6,
     &           /T2,'OIL DSMAX (',I3,')',T28,'=',E13.6,T43,
     &                 'WATER DSMAX (',I3,')',T69,'=',E13.6,T84,
     &           /T2,'OIL MATERIAL BALANCE (%)',T28,'=',E13.6,T43,
     &                 'WATER MATERIAL BALANCE (%)',T69,'=',E13.6//)
      WRITE(IOCODE,3040)OPR,COP,WPR,CWP,OIR,COI,WIR,CWI,WORP,CWORP,
     &                            QPT,CQPT,DIFRAT,DIFCUM
 3040 FORMAT(/T2,'OIL PRODUCTION RATE (m**3/s)',T35,'=',E15.6,
     &           T65,'CUM. OIL PRODUCTION (m**3)',T95,'=',E15.6,
     &            /T2,'WATER PRODUCTION RATE (m**3/s)',T35,'=',E15.6,
     &             T65,'CUM. WATER PRODUCTION (m**3)',T95,'=',E15.6,
     &           //T2,'OIL INJECTION RATE (m**3/s)',T35,'=',E15.6,
     &             T65,'CUM. OIL INJECTION (m**3)',T95,'=',E15.6,
     &            /T2,'WATER INJECTION RATE (m**3/s)',T35,'=',E15.6,
     &             T65,'CUM. WATER INJECTION (m**3)',T95,'=',E15.6,
     &           //T2,'SUB-CUM. WOR (m**3/m**3)',T35,'=',E15.6,
     &             T65,'CUMULATIVE WOR (m**3/m**3)',T95,'=',E15.6,
     &           //T2,'TOTAL PRODUCTION RATE (m**3/s)',T35,'=',E15.6,
     &             T65,'(/DT) TOTAL PRODUCTION (m**3)',T95,'=',E15.6,
     &           T115,'(FROM PAR. DIFF. EQ.)',
     &            /T2,'DIFF. IN TOT. PROD. RATE (m**3/s)',T35,'=',E15.6,
     &             T65,'DIFF. IN TOT. PROD. (m**3)',T95,'=',E15.6//)
C
   60 IF(NLOOP.EQ.1.AND.K.NE.1)GO TO 120
      IF(NLOOP.EQ.1)WRITE(IOCODE,3050)
 3050 FORMAT(/T15,7('*'),'INITIAL ARRAYS',7('*')/)
      IF(IPOMAP.EQ.0.AND.NLOOP.NE.1)GO TO 70
      WRITE(IOCODE,3060)
 3060 FORMAT(/T15,'******  OIL PRESSURE DISTRIBUTION   ******')
      WRITE(IOCODE,3070)(PO(I,K),I=1,II)
 3070 FORMAT(2X,8E15.6)
   70 IF(IPWMAP.EQ.0.AND.NLOOP.NE.1)GO TO 80
      WRITE(IOCODE,3080)
 3080 FORMAT(/T15,'*****  WATER PRESSURE DISTRIBUTION  *****')
      WRITE(IOCODE,3070)(PW(I,K),I=1,II)
   80 IF(IPCMAP.EQ.0.AND.NLOOP.NE.1)GO TO 90
      WRITE(IOCODE,3090)
 3090 FORMAT(/T15,'*****  CAPILLARY PRESSURE DISTRIBUTION  **'
     &          ,'***')
      WRITE(IOCODE,3070)(PC(I,K),I=1,II)
   90 IF(ISOMAP.EQ.0.AND.NLOOP.NE.1)GO TO 100
      WRITE(IOCODE,3100)
 3100 FORMAT(/T15,'************  OIL SATURATION  ************')
      WRITE(IOCODE,3070)(SO(I,K),I=1,II)
  100  IF(ISWMAP.EQ.0.AND.NLOOP.NE.1)GO TO 110
      WRITE(IOCODE,3110)
 3110 FORMAT(/T15,'***********  WATER SATURATION  ***********')
      WRITE(IOCODE,3070)(SW(I,K),I=1,II)
  110  IF(NLOOP.NE.1) WRITE(IOCODE,3120)
 3120 FORMAT(/T5,54('*'),'      END OF REPORT       ',54('*'),/)
  120  RETURN
      END
C
C.........................................................SWSOIR
C
      SUBROUTINE SWSOIR(II)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/PORPER/POR(30),PERMX(30),TX(29)
      COMMON/SWSO/SWIR(30),SOR(30)
      COMMON/CONST/AC,RF
      DO 10 I=1,II
      AK=PERMX(I)*1.01325D+15
      SWIR(I)=100.0D0*POR(I)**2.25D0/DSQRT(AK)
      SWIR1=1.0D0-SWIR(I)
      SOR(I)=SWIR1-RF*SWIR1
   10 CONTINUE
      RETURN
      END
C
C...........................................................TABLE
C
      SUBROUTINE TABLE(IICODE,IOCODE)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/TAB/SAT(25),AKROT(25),AKRWT(25),PCOWT(25),MSAT
```

```
         DIMENSION IHEDIN(80)
C
C*****  RELATIVE PERMEABILITY AND CAPILLARY PRESSURE TABLE
C
         WRITE(IOCODE,3000)
  3000   FORMAT(//T15,'**********   EMPIRICAL DATA TABLE   **********')
         READ(IICODE,2000)(IHEDIN(IH),IH=1,40)
  2000   FORMAT(40A2)
         WRITE(IOCODE,3010)(IHEDIN(IH),IH=1,40)
  3010   FORMAT(//,1X,40A2/)
         DO 10 I=1,25
         READ(IICODE,2010)SAT(I),AKROT(I),AKRWT(I),PCOWT(I)
         WRITE(IOCODE,2010)SAT(I),AKROT(I),AKRWT(I),PCOWT(I)
         IF(SAT(I).GE.1.10D0)GO TO 20
    10   CONTINUE
    20   MSAT=I
  2010   FORMAT(4E15.6)
         RETURN
         END
C
C.............................................................THOMAS
C
         SUBROUTINE THOMAS(A,B,C,D,X,N)
         IMPLICIT REAL*8(A-H,O-Z)
         DIMENSION A(1),B(1),C(1),D(1),X(1),Q(1001),G(1001)
C
C*****  SOLUTION OF TRIDIAGONAL SYSTEM OF EQUATIONS
C         A-MAIN DIAGONAL
C         B-SUPERDIAGONAL
C         C-SUBDIAGONAL
C         D-R.H.S. VECTOR
C         X-SOLUTION VECTOR
C         N-ORDER OF MATRIX (NUMBER OF UNKNOWNS)
C
         WI=A(1)
         G(1)=D(1)/WI
         DO 10 I=2,N
         Q(I-1)=B(I-1)/WI
         WI=A(I)-C(I)*Q(I-1)
    10   G(I)=(D(I)-C(I)*G(I-1))/WI
         X(N)=G(N)
         DO 20 I=2,N
         J=N-I+1
    20   X(J)=G(J)-Q(J)*X(J+1)
         RETURN
         END
C
C.............................................................TIME
C
         SUBROUTINE TIME(IICODE,IOCODE,NN,FACT1,FACT2,TMAX,DSMAX,
        *                DPMAX)
         IMPLICIT REAL*8(A-H,O-Z)
         DIMENSION IHEDIN(80)
C
         READ(IICODE,2000)(IHEDIN(IH),IH=1,40)
  2000   FORMAT(40A2)
         READ(IICODE,2010)NN,FACT1,FACT2,TMAX
  2010   FORMAT(I5,3E15.6)
         WRITE(IOCODE,3000)NN,FACT1,FACT2,TMAX
  3000   FORMAT(//T15,'MAXIMUM NUMBER OF TIME STEPS   =',I5,
        &          /T15,'FACTOR FOR INCREASING DELT      =',E15.6,3X,
        &                'WHEN DSMAX AND DPMAX ARE NOT EXCEEDED.',
        &          /T15,'FACTOR FOR DECREASING DELT      =',E15.6,3X,
        &                'WHEN DSMAX OR DPMAX IS EXCEEDED.',
        &          /T15,'MAXIMUM SIMULATION TIME         =',E15.6,2X,'S'//)
```

```
C
C   NN      -- MAXIMUM NUMBER OF TIME STEP
C   FACT1 -- FACTOR FOR INCREASING TIME-STEP SIZE
C   FACT2 -- FACTOR FOR DECREASING TIME-STEP SIZE
C   TMAX   -- MAXIMUM SUMULATION TIME
      READ(IICODE,2000)(IHEDIN(IH),IH=1,40)
      READ(IICODE,2020)DSMAX,DPMAX
 2020 FORMAT(2E15.6)
      WRITE(IOCODE,3010)DSMAX,DPMAX
 3010 FORMAT(//T15,'AUTOMATIC TIME STEP CRITERIA:',
     &          /T20,'MAXIMUM ALLOWED SATURATION CHANGE (DSMAX)   = ',
     &   E15.6/T20,'MAXIMUM ALLOWED PRESSURE CHANGE (DPMAX)      = ',
     &   E15.6/)
      RETURN
      END
C
C............................................................TRAN
C
      SUBROUTINE TRAN(II)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/SIZE/DELX(30),DELXC,DELY,DELZ,DEPTH(30)
      COMMON/PORPER/POR(30),PERMX(30),TX(29)
C
C***** FOR EQUAL BLOCK SIZE ONLY (EXCEPT AT THE INLET AND OUTLET)
C
      AREA=DELY*DELZ*2.0D0
      IIM1=II-1
      IIM2=II-2
      TX(1)=AREA*PERMX(1)/DELXC
      TX(IIM1)=AREA*PERMX(II)/DELXC
      DO 10 I=2,IIM2
      TX(I)=AREA*PERMX(I)*PERMX(I+1)/(PERMX(I)+PERMX(I+1))/DELXC
   10 CONTINUE
      RETURN
      END
```